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Accelerated forward-backward algorithm based on inertial and correction terms with linesearch for solving convex minimization problem and its application

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ABSTRACT. This paper presents an accelerated variant of the proximal forward-backward splitting method designed for solving convex minimization problem in Hilbert spaces. Our proposed algorithm integrates an inertial extrapolation term and two additional correction terms, coupled with linesearch stepsize that circumvents the explicit need for Lipschitz constant estimation. We establish weak convergence theorem, demonstrating that our method approximates solutions to convex minimization problems. Numerical experiments confirm the practical effectiveness and accelerated convergence speed of our algorithm, particularly highlighting its application in image recovery problem.

1. INTRODUCTION

The primary objective of this paper is to investigate the convergence properties of a variant of the proximal forward- backward splitting method for solving the following convex minimization problem:

$$\min_{x\in H} f(x) + g(x), \tag{1.1}$$

where H is real Hilbert space, and $g : H \to (-\infty, +\infty]$ is proper, lower semicontinuous and convex and $f : H \to R$ is convex and frèchet differentiable function with the Lipschitz continuous gradient denoted by ∇f. Utilizing Fermat's rule, problem (1.1) translates into locating $x^* \in H$ satisfying the first-order optimality condition:

$$0 \in (\partial g + \nabla f)(x^*), \tag{1.2}$$

where ∂g denotes the subdifferential of g. The optimal solution set of this problem (1.1) denoted by Ω. The convex minimization problem has attracted substantial interest from researchers in the optimization field due to its widespread and diverse applications in various disciplines such as optimal control, signal processing, system identification, machine learning, and image analysis (see, e.g., [6, 7, 8, 14]).

To describe and motivate methods, the proximal operator was defined by $prox_g = (\partial g + I)^{-1}$, where ∂g is the classical convex subdifferential of g and I is the identity operator in H. For any positive parameter λ, x is an optimal solution to the convex minimization problem if and only if $x = prox_{\lambda g}(x - \lambda\nabla f(x))$. In this connection, we can define a simple splitting method

$$x_{n+1} = prox_{\lambda_n g}(x_n - \lambda_n \nabla f(x_n)), \tag{1.3}$$

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with positive stepsize λ_n. This method is often called the forward–backward algorithm.The convergence of the iteration (1.3) to an optimal solution of (1.1) is usually established under the assumption that the gradient of f is Lipschitz continuous and the stepsize λ_n is taken bounded below and less than some constant related to the Lipschitz constant. However, when the gradient ∇f exhibits Lipschitz continuity but the explicit Lipschitz constant is unknown or difficult to estimate accurately, determining an appropriate stepsize λ_n that guarantees convergence becomes particularly challenging.

In 2016, Cruz and Nghia [3] addressed this issue by introducing linesearch strategy that avoids the explicit Lipschitz continuity requirement. The stepsize generated by both linesearch converge to zero and the complexity $o(k^{-1})$ in terms of cost function evaluations. Several researchers have actively explored and enhanced forward-backward iterations, focusing on developing innovative stepsize rules and convergence theorems aimed at accelerating convergence performance: see, e.g. [2, 5, 6, 9, 10, 12, 13].

In 2024, Izuchukwu et al. [11] proposed a proximal point algorithm to solve the monotone inclusion problem in Hilbert spaces. Their algorithm integrates the classical proximal point method with an inertial extrapolation term and two correction terms. They further illustrated through numerical experiments that their method performed efficiently compared with related algorithms. The proposed method is explicitly given as:

$$\begin{aligned} w_n &= x_n + \alpha(x_n - x_{n-1}) + \delta(1+\alpha)(w_{n-1} - x_n) - \alpha\delta(w_{n-2} - x_{n-1}), \\ y_n &= (I - \partial g)^{-1}(w_n), n \geq 0. \end{aligned} \tag{1.4}$$

The proxinal point algorithm contains inertial extrapolation tern $\alpha(x_n - x_{n-1})$ and two correction term $\delta(1+\alpha)(w_{n-1} - x_n)$ and $\alpha\delta(w_{n-2} - x_{n-1})$. Both the inertial extrapolation term and the two correction term was added to the original proximal point algorithm in order to improve on the numerical convergence speed of algorithm.

Motivated by the fast proximal point algorithm of Izuchukwu et al. [11], the work of Cruz and Nghia [3], we propose the forward-backward algorithm with combining of inertial term and two correction terms. Our method employs linesearch stepsize which do not need the Lipschitz constants of the associated operators for its execution. Using algorithm, we prove weak convergence theorem for approximating a solution to the convex minimization problem.

The rest of the paper is organized as follows: In Section 2, we introduce some important definitions and lemmas which are useful in establishing our main results. In Section 3, we introduce our proposed algorithm and then prove its convergence analysis. By presenting numerical examples in Section 4, we demonstrate the practicality of our algorithm to image recovery. Finally, in Section 5, we give a conclusion of our paper.

2. PRELIMINARIES

In this section, we present key definitions and auxiliary results that support the convergence analysis of our proposed algorithm.

Let $\{x_n\} \subset \Omega$ be a sequence in a Hilbert space H. We denote weak convergence to $z \in \Omega$ by $x_n \rightharpoonup z$, and strong convergence by $x_n \to z$.

A mapping $T : H \to H$ is said to be Lipschitz continuous with constant $L > 0$ if it satisfies the inequality

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in H.$$

Let $g : H \to (-\infty, +\infty]$ be a proper, convex, and lower semicontinuous function. Its domain is given by

$$\mathrm{dom} g = \{x \in H | g(x) < +\infty\}.$$

For any $x \in \mathrm{dom}g$, the subdifferential of g at x is defined by

$$\partial g(x) = \{v \in H | \langle v, y - x\rangle \leq g(y) - g(x),\ y \in H\}.$$

The proximal operator of g, denoted by $\mathrm{prox}_g : \mathrm{dom}g \to H$, is defined as

$$\mathrm{prox}_g(z) = (I + \partial g)^{-1}(z), \text{for } z \in H.$$

This operator is known to be single-valued and satisfies the inclusion:

$$\frac{z - \mathrm{prox}_{\lambda g}(z)}{\lambda} \in \partial g(\mathrm{prox}_{\lambda g}(z)) \quad \text{for all } z \in H,\ \lambda > 0. \tag{2.5}$$

Definition 2.1. *Let $S \subset H$ be a nonempty set. A sequence $\{x_n\} \subset H$ is said to be quasi-Fejér convergent to S if, for every $x \in S$, there exists a nonnegative sequence $\{\varepsilon_n\}$ with $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$ such that*

$$\|x_{n+1} - x\|^2 \leq \|x_n - x\|^2 + \varepsilon_n\ \forall n \geq 1.$$

If $\{\varepsilon_n\}$ is a null sequence, then $\{x_n\}$ is said to be Fejér convergent to S.

Lemma 2.1. [4] *The subdifferential operator ∂g is maximal monotone. Moreover, its graph*

$$\mathrm{Gph}(\partial g) = \{(x, v) \in H \times H : v \in \partial g(x)\}$$

is demiclosed, meaning that if $\{(x_n, v_n)\} \subset \mathrm{Gph}(\partial g)$, $x_n \rightharpoonup x$, and $v_n \to v$, then $(x, v) \in \mathrm{Gph}(\partial g)$

Lemma 2.2. [1, 10] *Let $\{x_n\} \subset H$ be a sequence that is quasi-Fejér convergent to a nonempty set S. Then:*

(i) The sequence $\{x_n\}$ is bounded.

(ii) If all weak limit points of $\{x_n\}$ belong to S, then $\{x_n\}$ converges weakly to a point in S.

3. Main results

In this section, we propose an accelerated algorithm to solve the convex minimization problem defined in (1.1). To proceed, we adopt the following assumptions on the objective functions f and g:

- **(I)** The functions $f, g : H \to (-\infty, +\infty]$ are proper, convex, and lower semicontinuous, with the domain inclusion $\mathrm{dom}g \subseteq \mathrm{dom}f$.
- **(II)** The function f is Fréchet differentiable on an open set that contains $\mathrm{dom}g$. Moreover, its gradient ∇f is uniformly continuous on bounded subsets of $\mathrm{dom}g$ and maps bounded sets into bounded sets in H

Based on these conditions, we now introduce a novel forward-backward iterative scheme that incorporates both inertial and correction terms to enhance convergence. The proposed algorithm aims to accelerate the forward-backward approach, particularly in scenarios where exact Lipschitz constants are unavailable or difficult to estimate.

Algorithm 3.1. Initialization: *Let $w_0, w_{-1}, x_0, x_1 \in H$, $\sigma > 0$, $\rho \in (0, 1)$, $\mu \in (0, 1)$, $\alpha_n \in (-1, 0]$ and $\delta \in [0, 1)$.*

Iterative step: *For $n \geq 1$, calculate x_{n+1} as follows:*

Step 1. *Compute the inertial and two correction step:*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) + \delta(1 + \alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}). \tag{3.6}$$

Step 2. *Compute the forward-backward step:*

$$y_n = prox_{\lambda_n g}(w_n - \lambda_n \nabla f(w_n)),$$

where $\lambda_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\lambda_n(\|\nabla f(prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n))) - \nabla f(y_n)\| + \|\nabla f(w_n) - \nabla f(y_n)\|)$$
$$\leq \frac{\mu}{2}(\|prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n)) - y_n\| + \|w_n - y_n\|).$$

If $y_n = w_n$, then stop. Otherwise, go to Step3.

Step 3. *Compute the x_{n+1} step:*

$$x_{n+1} = prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n)).$$

If $x_{n+1} = y_n$, then stop. Otherwise, set $n = n + 1$ and go back to Step1.

Remark 3.1. *The condition that $\{\lambda_n\}$ is bounded away from 0 can be removed when ∇f is Lipschitz continuous. Indeed, if ∇f is L-Lipschitz continuous on H, then $\lambda_n \leq \sigma$, $n \geq 1$. If $\lambda_n < \sigma$, then $\lambda_n = \sigma\rho^{m_n}$ where m_n is the smallest positive integer such that*

$$\lambda_n(\|\nabla f(prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n))) - \nabla f(y_n)\| + \|\nabla f(w_n) - \nabla f(y_n)\|)$$
$$\leq \frac{\mu}{2}(\|prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n)) - y_n\| + \|w_n - y_n\|). \tag{3.7}$$

Set $\widehat{\lambda_n} = \frac{\lambda_n}{\rho}$. By the Lipschitz continuity of ∇f and (3.7), we have

$$\begin{aligned} & \widehat{\lambda_n} L(\|prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n)) - y_n\| + \|w_n - y_n\|) \\ \geq\ & \widehat{\lambda_n}(\|\nabla f(prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n))) - \nabla f(y_n)\| + \|\nabla f(w_n) - \nabla f(y_n)\|) \\ >\ & \frac{\mu}{2}(\|prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n)) - y_n\| + \|w_n - y_n\|). \end{aligned}$$

So $\widehat{\lambda_n} L > \frac{\mu}{2}$. It follows that $\lambda_n > \frac{\mu\rho}{2L}$. Hence $\lambda_n \geq \min\{\sigma, \frac{\mu\rho}{2L}\}$ for all $n \geq 1$.

Theorem 3.1. *Let $\{x_n\}$ be generated by Algorithm 3.1 and $\lambda_n \geq \lambda$ for some $\lambda > 0$. Then the sequence $\{x_n\}$ weakly converges to an element of Ω.*

Proof. By definition of proximal operator, we have

$$\frac{w_n - y_n}{\lambda_n} - \nabla f(w_n) = \frac{w_n - prox_{\lambda_n g}(w_n - \lambda_n \nabla f(w_n))}{\lambda_n} - \nabla f(w_n) \in \partial g(y_n).$$

It follows from the convexity of g that

$$g(x) - g(y_n) \geq \langle \frac{w_n - y_n}{\lambda_n} - \nabla f(w_n), x - y_n \rangle, \forall x \in \mathrm{dom} g. \tag{3.8}$$

Also, we have

$$\frac{y_n - x_{n+1}}{\lambda_n} - \nabla f(y_n) = \frac{y_n - prox_{\lambda_n g}(y_n - \lambda_n \nabla f(y_n))}{\lambda_n} - \nabla f(y_n) \in \partial g(x_{n+1}).$$

By the convexity of g, we also have

$$g(x) - g(x_{n+1}) \geq \langle \frac{y_n - x_{n+1}}{\lambda_n} - \nabla f(y_n), x - x_{n+1} \rangle, \forall x \in \mathrm{dom} g. \tag{3.9}$$

On the other hand, for any $x \in \mathrm{dom} g \subseteq \mathrm{dom} f$, we have

$$f(x) - f(w_n) \geq \langle \nabla f(w_n), x - w_n \rangle \tag{3.10}$$

and

$$f(x) - f(y_n) \geq \langle \nabla f(y_n), x - y_n \rangle. \tag{3.11}$$

Combining (3.8), (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}
& g(x)-g(x_{n+1})+g(x)-g(y_n)+f(x)-f(y_n)+f(x)-f(w_n)\\
\geq\ & \Big\langle \frac{y_n-x_{n+1}}{\lambda_n}-\nabla f(y_n), x-x_{n+1}\Big\rangle+\Big\langle \frac{w_n-y_n}{\lambda_n}-\nabla f(w_n), x-y_n\Big\rangle\\
& +\langle \nabla f(y_n), x-y_n\rangle+\langle \nabla f(w_n), x-w_n\rangle\\
=\ & \frac{1}{\lambda_n}\langle y_n-x_{n+1}, x-x_{n+1}\rangle+\frac{1}{\lambda_n}\langle w_n-y_n, x-y_n\rangle+\langle \nabla f(y_n), x_{n+1}-y_n\rangle\\
& +\langle \nabla f(w_n), y_n-w_n\rangle\\
=\ & \frac{1}{\lambda_n}\langle y_n-x_{n+1}, x-x_{n+1}\rangle+\frac{1}{\lambda_n}\langle w_n-y_n, x-y_n\rangle+\langle \nabla f(y_n)-\nabla f(x_{n+1}), x_{n+1}-y_n\rangle\\
& +\langle \nabla f(x_{n+1}), x_{n+1}-y_n\rangle+\langle \nabla f(w_n)-\nabla f(y_n), y_n-w_n\rangle+\langle \nabla f(y_n), y_n-w_n\rangle\\
\geq\ & \frac{1}{\lambda_n}[\langle y_n-x_{n+1}, x-x_{n+1}\rangle+\langle w_n-y_n, x-y_n\rangle]-\|\nabla f(y_n)-\nabla f(x_{n+1})\|\|x_{n+1}-y_n\|\\
& -\|\nabla f(w_n)-\nabla f(y_n)\|\|y_n-w_n\|+f(x_{n+1})-f(y_n)+f(y_n)-f(w_n)\\
\geq\ & \frac{1}{\lambda_n}[\langle y_n-x_{n+1}, x-x_{n+1}\rangle+\langle w_n-y_n, x-y_n\rangle]\\
& -\|\nabla f(y_n)-\nabla f(x_{n+1})\|(\|y_n-w_n\|+\|x_{n+1}-y_n\|)\\
& -\|\nabla f(w_n)-\nabla f(y_n)\|(\|y_n-w_n\|+\|x_{n+1}-y_n\|)+f(x_{n+1})-f(w_n)\\
=\ & \frac{1}{\lambda_n}[\langle y_n-x_{n+1}, x-x_{n+1}\rangle+\langle w_n-y_n, x-y_n\rangle]\\
& -(\|\nabla f(y_n)-\nabla f(x_{n+1})\|+\|\nabla f(w_n)-\nabla f(y_n)\|)(\|y_n-w_n\|+\|x_{n+1}-y_n\|)\\
& +f(x_{n+1})-f(w_n).
\end{aligned}$$

It follows that

$$\begin{aligned}
(3.12)\quad & \frac{1}{\lambda_n}[\langle y_n-x_{n+1}, x_{n+1}-x\rangle+\langle w_n-y_n, y_n-x\rangle]\\
\geq\ & g(x_{n+1})-g(x)+g(y_n)-g(x)+f(y_n)-f(x)+f(w_n)-f(x)+f(x_{n+1})-f(w_n)\\
& -(\|\nabla f(y_n)-\nabla f(x_{n+1})\|+\|\nabla f(w_n)-\nabla f(y_n)\|)(\|y_n-w_n\|+\|x_{n+1}-y_n\|).
\end{aligned}$$

From (3.12) and condition of λ_n, it follows that

$$\begin{aligned}
& \langle y_n-x_{n+1}, x_{n+1}-x\rangle+\langle w_n-y_n, y_n-x\rangle\\
\geq\ & \lambda_n[(f+g)(x_{n+1})-(f+g)(x)+(f+g)(y_n)-(f+g)(x)]\\
& -\lambda_n(\|\nabla f(y_n)-\nabla f(x_{n+1})\|+\|\nabla f(w_n)-\nabla f(y_n)\|)(\|y_n-w_n\|+\|x_{n+1}-y_n\|)\\
\geq\ & \lambda_n[(f+g)(x_{n+1})-(f+g)(x)+(f+g)(y_n)-(f+g)(x)]\\
& -\frac{\mu}{2}(\|x_{n+1}-y_n\|+\|w_n-y_n\|)(\|y_n-w_n\|+\|x_{n+1}-y_n\|)\\
\geq\ & \lambda_n[(f+g)(x_{n+1})-(f+g)(x)+(f+g)(y_n)-(f+g)(x)]\\
(3.13)\quad & -\frac{\mu}{2}(\|x_{n+1}-y_n\|^2+\|w_n-y_n\|^2).
\end{aligned}$$

We know that

$$(3.14)\qquad 2\langle y_n-x_{n+1}, x_{n+1}-x\rangle=\|y_n-x\|^2-\|y_n-x_{n+1}\|^2-\|x_{n+1}-x\|^2$$

and

$$(3.15)\qquad 2\langle w_n-y_n, y_n-x\rangle=\|w_n-x\|^2-\|w_n-y_n\|^2-\|y_n-x\|^2.$$

From (3.13), (3.14) and (3.15), we get

$$\begin{aligned} & -\frac{1}{2}\|y_n - x_{n+1}\|^2 - \frac{1}{2}\|x_{n+1} - x\|^2 + \frac{1}{2}\|w_n - x\|^2 - \frac{1}{2}\|w_n - y_n\|^2 \\ \geq\ & \lambda_n[(f+g)(x_{n+1}) - (f+g)(x) + (f+g)(y_n) - (f+g)(x)] \\ & -\frac{\mu}{2}(\|x_{n+1} - y_n\|^2 + \|w_n - y_n\|^2). \end{aligned}$$

This shows that

$$\begin{aligned} & \|w_n - x\|^2 - \|x_{n+1} - x\|^2 \\ \geq\ & 2\lambda_n[(f+g)(x_{n+1}) - (f+g)(x) + (f+g)(y_n) - (f+g)(x)] \\ & +(1-\mu)\|x_{n+1} - y_n\|^2 + (1-\mu)\|w_n - y_n\|^2. \end{aligned} \tag{3.16}$$

Setting $x = x^*$, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq\ & \|w_n - x^*\|^2 - 2\lambda_n[(f+g)(x_{n+1}) - (f+g)(x^*) \\ & +(f+g)(y_n) - (f+g)(x^*)] - (1-\mu)\|x_{n+1} - y_n\|^2 \\ & -(1-\mu)\|w_n - y_n\|^2. \end{aligned} \tag{3.17}$$

We define

$$z_n = x_n + \delta(w_{n-1} - x_n), \tag{3.18}$$

$$z_n = (1-\delta)x_n + \delta w_{n-1}$$

$$z_n - \delta w_{n-1} = (1-\delta)x_n, \tag{3.19}$$

it follows that

$$x_n = \frac{1}{1-\delta}z_n - \frac{\delta}{1-\delta}w_{n-1}. \tag{3.20}$$

From (3.18) and (3.20), we consider

$$\begin{aligned} x_n - z_n &= \frac{1}{1-\delta}z_n - \frac{\delta}{1-\delta}w_{n-1} - z_n \\ &= \frac{1}{1-\delta}[z_n - \delta w_{n-1} - (1-\delta)z_n] \\ &= \frac{\delta}{1-\delta}(z_n - w_{n-1}) \\ &= \frac{\delta}{1-\delta}[(x_n + \delta(w_{n-1} - x_n)) - w_{n-1}] \\ &= \frac{\delta}{1-\delta}[(1-\delta)x_n - (1-\delta)w_{n-1}] \\ &= \frac{\delta}{(1-\delta)}(1-\delta)[x_n - w_{n-1}] \\ &= \delta(x_n - w_{n-1}). \end{aligned} \tag{3.21}$$

Consider,

$$\frac{\delta}{1-\delta}(z_n - w_{n-1}) = \delta(x_n - w_{n-1}),$$

it follows that

$$\|z_n - w_{n-1}\|^2 = (1-\delta)^2\|(x_n - w_{n-1})\|^2. \tag{3.22}$$

From (3.20) and (3.22), we get

$$\begin{aligned}
\|x_n - x^*\|^2 &= \|(\frac{1}{1-\delta}z_n - \frac{\delta}{1-\delta}w_{n-1}) - x^*\|^2 \\
&= \|\frac{1}{1-\delta}(z_n - x^*) - \frac{\delta}{1-\delta}(w_{n-1} - x^*)\|^2 \\
&= \frac{1}{1-\delta}\|z_n - x^*\|^2 - \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 + \frac{\delta}{(1-\delta)^2}\|z_n - w_{n-1}\|^2 \\
&= \frac{1}{1-\delta}\|z_n - x^*\|^2 - \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 + \delta\|x_n - w_{n-1}\|^2. \qquad (3.23)
\end{aligned}$$

From (3.18), we have

$$z_{n-1} = x_{n-1} + \delta(w_{n-2} - x_{n-1}). \qquad (3.24)$$

By definition of the sequence w_n, we obtain

$$\begin{aligned}
w_n &= x_n + \alpha_n(x_n - x_{n-1}) + \delta(1+\alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}) \\
&= x_n + \alpha_n(x_n - x_{n-1}) + \delta(w_{n-1} - x_n) + \delta\alpha_n(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}) \\
&= x_n + \delta(w_{n-1} - x_n) + \alpha_n[(x_n - x_{n-1}) + \delta(w_{n-1} - x_n) - \delta(w_{n-2} - x_{n-1})] \\
&= x_n + \delta(w_{n-1} - x_n) + \alpha_n[(x_n + \delta(w_{n-1} - x_n)) - (x_{n-1} + \delta(w_{n-2} - x_{n-1}))]. \qquad (3.25)
\end{aligned}$$

Replacing (3.18) and (3.24) in (3.25), we obtain

$$w_n = z_n + \alpha_n(z_n - z_{n-1}). \qquad (3.26)$$

From (3.26), we obtain

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|z_n + \alpha_n(z_n - z_{n-1}) - x^*\|^2 \\
&= \|z_n + \alpha_n z_n - \alpha_n z_{n-1} - x^*\|^2 \\
&= \|(1+\alpha_n)(z_n - x^*) - \alpha_n(z_{n-1} - x^*)\|^2 \\
&= (1+\alpha_n)\|z_n - x^*\|^2 - \alpha_n\|z_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|z_n - z_{n-1}\|^2 \qquad (3.27)
\end{aligned}$$

From (3.17) and (3.27)

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq\ & (1+\alpha_n)\|z_n - x^*\|^2 - \alpha_n\|z_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|z_n - z_{n-1}\|^2 \\
& -2\lambda_n[(f+g)(x_{n+1}) - (f+g)(x^*) \\
& +(f+g)(y_n) - (f+g)(x^*)] - (1-\mu)\|x_{n+1} - y_n\|^2 \\
& -(1-\mu)\|w_n - y_n\|^2. \qquad (3.28)
\end{aligned}$$

From (3.23) and (3.16), we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \\
\leq\ & \|w_n - x^*\|^2 - \|x_n - x^*\|^2 - 2\lambda_n[(f+g)(x_{n+1}) - (f+g)(x^*) + (f+g)(y_n) \\
& -(f+g)(x^*)] + (1-\mu)\|x_{n+1} - y_n\|^2 + (1-\mu)\|w_n - y_n\|^2 \\
=\ & \|w_n - x^*\|^2 - \frac{1}{1-\delta}\|z_n - x^*\|^2 + \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 - \delta\|x_n - w_{n-1}\|^2 \\
& -2\lambda_n[(f+g)(x_{n+1}) - (f+g)(x^*) + (f+g)(y_n) - (f+g)(x^*)] \\
& -(1-\mu)\|x_{n+1} - y_n\|^2 - (1-\mu)\|w_n - y_n\|^2. \qquad (3.29)
\end{aligned}$$

From (3.27) and (3.29)

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-\|x_n-x^*\|^2\\
\leq\ &(1+\alpha_n)\|z_n-x^*\|^2-\alpha_n\|z_{n-1}-x^*\|^2+\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
&-\frac{1}{1-\delta}\|z_n-x^*\|^2+\frac{\delta}{1-\delta}\|w_{n-1}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)+(f+g)(y_n)-(f+g)(x^*)]\\
&+(1-\mu)\|x_{n+1}-y_n\|^2+(1-\mu)\|w_n-y_n\|^2\\
=\ &(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\alpha_n\|z_{n-1}-x^*\|^2+\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
&+\frac{\delta}{1-\delta}\|w_{n-1}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)+(f+g)(y_n)-(f+g)(x^*)]\\
&-(1-\mu)\|x_{n+1}-y_n\|^2-(1-\mu)\|w_n-y_n\|^2. \qquad (3.30)
\end{aligned}$$

Again, from (3.27) and (3.30), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-\|x_n-x^*\|^2\\
\leq\ &(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\alpha_n\|z_{n-1}-x^*\|^2+\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
&+\frac{\delta}{1-\delta}[(1+\alpha_{n-1})\|z_{n-1}-x^*\|^2-\alpha_{n-1}\|z_{n-2}-x^*\|^2+\alpha_{n-1}(1+\alpha_{n-1})\|z_{n-1}-z_{n-2}\|^2]\\
&-\delta\|x_n-w_{n-1}\|^2-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)+(f+g)(y_n)-(f+g)(x^*)]\\
&-(1-\mu)\|x_{n+1}-y_n\|^2-(1-\mu)\|w_n-y_n\|^2. \qquad (3.31)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
\leq\ &\|x_n-x^*\|^2+\big(\frac{\delta(1+\alpha_{n-1})}{1-\delta}-\alpha_n\big)\|z_{n-1}-x^*\|^2+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2\\
&-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)+(f+g)(y_n)-(f+g)(x^*)]\\
&+(1-\mu)\|x_{n+1}-y_n\|^2+(1-\mu)\|w_n-y_n\|^2\\
=\ &\|x_n-x^*\|^2+\big(\frac{\delta}{1-\delta}+\frac{\delta(1+\alpha_{n-1})}{1-\delta}-\alpha_n\big)\|z_{n-1}-x^*\|^2\\
&+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)+(f+g)(y_n)-(f+g)(x^*)]\\
&-(1-\mu)\|x_{n+1}-y_n\|^2-(1-\mu)\|w_n-y_n\|^2. \qquad (3.32)
\end{aligned}$$

From (3.32), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-1}-x^*\|^2-\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
\leq\ &\|x_n-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)\\
&+(f+g)(y_n)-(f+g)(x^*)]-(1-\mu)\|x_{n+1}-y_n\|^2-(1-\mu)\|w_n-y_n\|^2.
\end{aligned}\tag{3.33}$$

Since $\alpha_{n-1}\leq\alpha_n$, it follows that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\frac{\delta\alpha_n}{1-\delta}\|z_{n-1}-x^*\|^2-\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
\leq\ &\|x_n-x^*\|^2-(\alpha_{n-1}-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)\\
&+(f+g)(y_n)-(f+g)(x^*)]-(1-\mu)\|x_{n+1}-y_n\|^2-(1-\mu)\|w_n-y_n\|^2\\
=\ &\|x_n-x^*\|^2-(\alpha_{n-1}-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2\\
&-\alpha_{n-1}(1+\alpha_{n-1})\|z_{n-1}-z_{n-2}\|^2\\
&+\frac{\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-\delta\|x_n-w_{n-1}\|^2\\
&-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)+(f+g)(y_n)-(f+g)(x^*)]\\
&-(1-\mu)\|x_{n+1}-y_n\|^2-(1-\mu)\|w_n-y_n\|^2.
\end{aligned}\tag{3.34}$$

We define,

$$\begin{aligned}
a_n\ =\ &\|x_n-x^*\|^2-(\alpha_{n-1}-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2\\
&-\alpha_{n-1}(1+\alpha_{n-1})\|z_{n-1}-z_{n-2}\|^2
\end{aligned}\tag{3.35}$$

From (3.34) and (3.35), we have

$$\begin{aligned}
a_{n+1}\ \leq\ &a_n+\frac{\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-\delta\|x_n-w_{n-1}\|^2\\
&-2\lambda_n[(f+g)(x_{n+1})-(f+g)(x^*)+(f+g)(y_n)-(f+g)(x^*)]\\
&-(1-\mu)\|x_{n+1}-y_n\|^2+(1-\mu)\|w_n-y_n\|^2.
\end{aligned}\tag{3.36}$$

We obtain that $a_n\geq 0$ since $\alpha_n\in(-1,0]$, $\delta\in[0,1)$ and $\mu\in(0,1)$. Furthermore, from (3.36), we have $\lim_{n\to\infty}a_n$ exists and $\{a_n\}$ is bounded. Hence, $\{x_n\}$ is bounded. From (3.36), we have

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0.\tag{3.37}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\hat{x}\in H$. Moreover, we obtain $x_{n_k+1}\rightharpoonup\hat{x}$. Since $\{x_{n_k}\}$ is bounded and $\|x_{n_k+1}-y_{n_k}\|\to 0$, by assumption (II), we obtain

$$\lim_{k\to\infty}\|\nabla f(x_{n_k+1})-\nabla f(y_{n_k})\|=0.\tag{3.38}$$

Note that

$$x_{n_k+1} = \text{prox}_{\lambda_{n_k} g}(y_{n_k} - \lambda_{n_k}\nabla f(y_{n_k})).$$

It follows from (2.5) that

$$\frac{y_{n_k} - \lambda_{n_k}\nabla f(y_{n_k}) - x_{n_k+1}}{\lambda_{n_k}} \in \partial g(x_{n_k+1}).$$

Hence

$$\frac{y_{n_k} - x_{n_k+1}}{\lambda_{n_k}} + \nabla f(x_{n_k+1}) - \nabla f(y_{n_k}) \in \nabla f(x_{n_k+1}) + \partial g(x_{n_k+1}).$$

Letting $k \to \infty$ in (3.39), from (3.38) and Lemma 2.1, we get

$$0 \in \partial(f+g)(\hat{x}). \tag{3.39}$$

Hence $\hat{x} \in \Omega$. By Lemma 2.2, we conclude that $\{x_n\}$ weakly converges to a point in Ω. This completes the proof. □

Special case: Let C be a nonempty closed convex subset of H. When g the indicator function of a closed convex set C, its proximal operator $prox_{\lambda g}$ reduces to projection onto C defined by $P_C x$. Consequently, we have the following results:

Algorithm 3.2. Initialization: *Let $w_0, w_{-1}, x_0, x_1 \in H$, $\sigma > 0$, $\rho \in (0,1)$, $\mu \in (0,1)$, $\alpha_n \in (-1,0]$ and $\delta \in [0,1)$.*
Iterative step: *For $n \geq 1$, calculate x_{n+1} as follows:*
Step 1. *Compute the inertial and two correction step:*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) + \delta(1+\alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}). \tag{3.40}$$

Step 2. *Compute the projection step:*

$$y_n = P_C(w_n - \lambda_n\nabla f(w_n)),$$

where $\lambda_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\lambda_n(\|\nabla f(P_C(y_n - \lambda_n\nabla f(y_n))) - \nabla f(y_n)\| + \|\nabla f(w_n) - \nabla f(y_n)\|)$$
$$\leq \frac{\mu}{2}(\|P_C(y_n - \lambda_n\nabla f(y_n)) - y_n\| + \|w_n - y_n\|).$$

If $y_n = w_n$, then stop. Otherwise, go to Step3.
Step 3. *Compute the x_{n+1} step:*

$$x_{n+1} = P_C(y_n - \lambda_n\nabla f(y_n)).$$

If $x_{n+1} = y_n$, then stop. Otherwise, set $n = n+1$ and go back to Step1.

From Theorem 3.1, we obtain corollary as the following:

Corollary 3.1. *Let $\{x_n\}$ be generated by Algorithm 3.2 and $\lambda_n \geq \lambda$ for some $\lambda > 0$. Then the sequence $\{x_n\}$ weakly converges to an element of the solution.*

4. Numerical experiment in image deblurring

We demonstrate the practical effectiveness of the proposed algorithm by applying it to an image deblurring problem, which is commonly formulated as a linear inverse problem:

$$Ax = b + v,$$

where $A \in \mathbb{R}^{N\times M}$ represents the blurring operator, $x \in \mathbb{R}^N$ is the original image, $b \in \mathbb{R}^M$ is the observed blurred image, and $v \in \mathbb{R}^M$ denotes the additive noise.

A standard approach to recovering the original image is to solve a regularized optimization problem, often posed as the well-known LASSO formulation [15]:

$$\min_{x\in\mathbb{R}^N}\left\{\frac{1}{2}\|b-Ax\|_2^2+\tau\|x\|_1\right\}, \tag{4.41}$$

where $\tau > 0$ is a regularization parameter, $\|\cdot\|_1$ denotes theℓ_1-norm, and $\|\cdot\|_2$ is the Euclidean norm.

The problem in (4.41) fits the composite minimization structure in (1.1), where we identify

$$f(x)=\frac{1}{2}\|b-Ax\|_2^2, \quad g(x)=\tau\|x\|_1.$$

To assess the performance of our proposed method, referred to as Algorithm 3.1, we compare it against two established approaches: the Fast Iterative Shrinkage-Thresholding Algorithm (FISTA) developed by Beck and Teboulle [2], and the linesearch-based forward-backward algorithm proposed by Cruz and Nghia [3].

The regularization parameters are chosen by $\tau = 10^{-5}$ and $w_0 = w_{-1} = x_0 = x_1 = (1, 1, 1, ..., 1) \in \mathbb{R}^N$.

For Algorithm 3.1, we set $\sigma = 2$, $\rho = 0.2$, $\mu = 0.3$, $\alpha_n = -\frac{1}{10(n+1)}$ and $\delta = 0.01$. We set $L = \frac{1}{\|A\|^2}$ in FISTA [2]. In Algorithm of Cruz and Nghia [3], we set $\sigma = 2$, $\theta = 0.2$ and $\delta = 0.3$.

For the experiments, we use RGB images of size 237×177, which are blurred using the following types of blur:

1. Motion blur with a length of 45 pixels and an orientation of $180°$.
2. Gaussian blur with a filter size of 5×5 and a standard deviation of 5.

We apply the Fast Fourier Transform (FFT) to convert the data into the frequency domain. To assess image quality, we use the Structural Similarity Index Measure (SSIM) [17] to evaluate the similarity between two images. The Peak Signal-to-Noise Ratio (PSNR), measured in decibels (dB), is defined as follows [16]:

$$PSNR = 10\log_{10}\left(\frac{255^2}{MSE}\right)$$

where MSE$= \|x_n - x\|^2$ and x is the original image. It is noted that, a higher PSNR value typically indicates better reconstruction quality. The resulting SSIM index ranges from 0 to 1, with a value of 1 indicating perfect structural similarity.

The numerical experiments were conducted in the MATLAB environment (version R2020b) on a MacBook Pro M1 with 8 GB of RAM. To recover the degraded RGB images, we limit the number of iterations to 500. The numerical results are reported as follows:

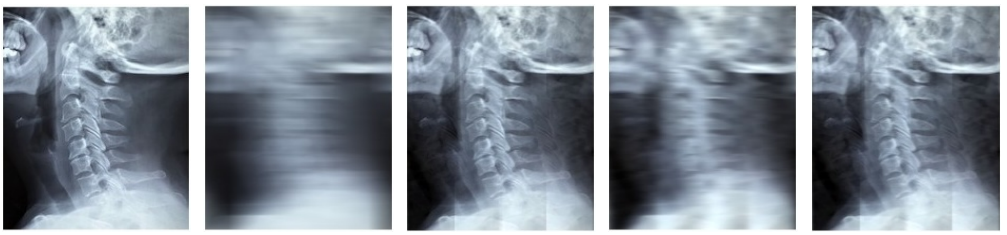
TABLE 1. The comparison of PSNR and SSIM for each algorithm of the restored images

Algorithms	Motion blur		Gaussian blur	
	PSNR	SSIM	PSNR	SSIM
Algorithm 3.1	29.8971	0.9010	39.9713	0.9771
FISTA [2]	23.7015	0.7193	31.7316	0.8988
Algorithm of Cruz and Nghia [3]	27.5375	0.8526	38.5324	0.9698

Table 1 shows that Algorithm 3.1 achieves a higher PSNR compared to both FISTA and the algorithm by Cruz and Nghia [3], under the same number of iterations. Furthermore,

the SSIM value obtained by Algorithm 3.1 is closer to 1, indicating stronger structural similarity. These results demonstrate the superior convergence performance of our proposed algorithm in this example.

We next show the restored images of RGB images for Motion blur.



(A) (B) (C) (D) (E)

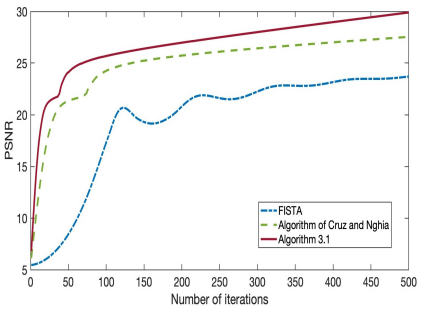
FIGURE 1. Figure (A) and (B) are the Original image and Motion blur image. Figure (C), (D) and (E) are the recovered images via Algorithm 3.1, FISTA [2] and Algorithm of Cruz and Nghia [3] for degraded images by Motion blur

We next show the restored images of RGB images for Gaussian blur.

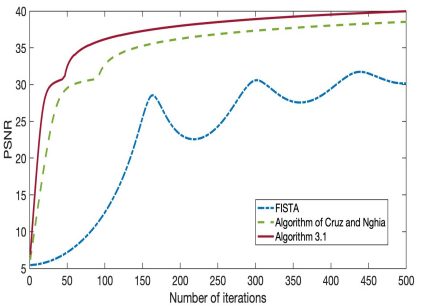


(A) (B) (C) (D) (E)

FIGURE 2. Figure (A) and (B) are the Original image and Gaussian blur image. Figure (C), (D) and (E) are the recovered images via Algorithm 3.1, FISTA [2] and Algorithm of Cruz and Nghia [3] for degraded images by Gaussian blur

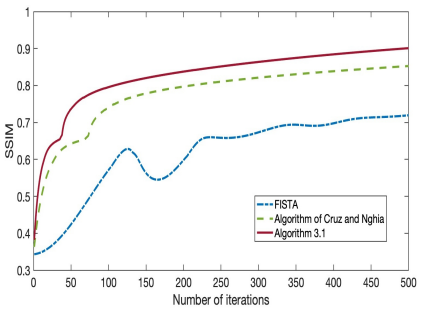


(A) PSNR plotting of Motion blur

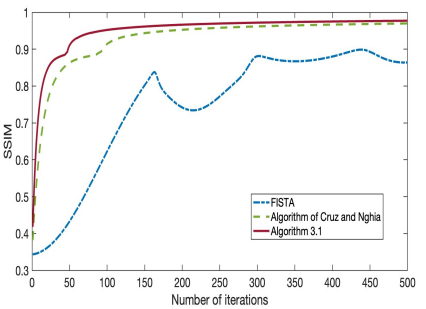


(B) PSNR plotting of Gaussian blur

FIGURE 3. Graphs of PSNR values for each blurred image and restored images by Algorithm 3.1, FISTA [2] and Algorithm of Cruz and Nghia [3]



(A) SSIM plotting of Motion blur



(B) SSIM plotting of Gaussian blur

FIGURE 4. Graphs of SSIM values for each blurred image and restored images by Algorithm 3.1, FISTA [2] and Algorithm of Cruz and Nghia [3]

The results of the numerical experiments are summarized in Table 1. Figures 1, 2, 3, and 4 present the recovered images and performance for each algorithm. These results demonstrate that Algorithm 3.1 outperforms both FISTA and the algorithm of Cruz and Nghia [3] in terms of PSNR and SSIM.

5. Conclusions

In this paper, we introduced a new accelerated forward-backward algorithm that integrates an inertial extrapolation term and two correction terms with linesearch technique for solving convex minimization problem. Our algorithm does not require the explicit knowledge of the Lipschitz constant of the gradient, making it more practical and flexible in real-world applications. We established a weak convergence theorem for the proposed method, ensuring its theoretical reliability. We demonstrated the effectiveness of our approach by applying it to image deblurring problem formulated as a LASSO model. Numerical experiments using RGB images degraded by Motion and Gaussian blur confirmed that our algorithm achieves better image reconstruction quality than FISTA and algorithm by Cruz and Nghia, as measured by PSNR and SSIM. These results highlight the potential of our algorithm for applications in image analysis.

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References

[1] Bauschke, Heinz H.; Combettes, Patrick L. Convex analysis and monotone operator theory in Hilbert spaces. With a foreword by Hédy Attouch. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. *Springer, New York*, 2011.

[2] Beck, Amir; Teboulle, Marc. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J. Imaging Sci.* **2** (2009), no. 1, 183–202.

[3] Bello Cruz, José Yunier; Nghia, Tran T. A. On the convergence of the forward-backward splitting method with linesearches. *Optim. Methods Softw.* **31** (2016), no. 6, 1209–1238.

[4] Burachik, Regina S.; Iusem, Alfredo N. Enlargements of monotone operators. In: *Set-Valued Mappings and Enlargements of Monotone Operators.* Boston, MA: Springer US, 2008, 161-220.

[5] Cholamjiak, Watcharaporn; Cholamjiak, Prasit; Suantai, Suthep. An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces. *J. Fixed Point Theory Appl.* **20** (2018), no. 1, Paper No. 42, 17 pp.

[6] Combettes, Patrick L.; Pesquet, Jean-Christophe. Proximal splitting methods in signal processing. In *Fixed-point algorithms for inverse problems in science and engineering*. New York, NY: Springer New York, 2011. p. 185-212.

[7] Combettes, Patrick L.; Wajs, Valérie R. Signal recovery by proximal forward-backward splitting. *Multiscale Model. Simul.* **4** (2005), no. 4, 1168–1200.

[8] Geman, Stuart; Geman, Donald. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Transactions on pattern analysis and machine intelligence*, (1984), 6: 721-741.

[9] Hanjing, Adisak; Suantai, Suthep. A fast image restoration algorithm based on a fixed point and optimization method. *Mathematics* **8**(3)(2020),378.

[10] Iusem, Alfredo N.; Svaiter, Benar Fux; Teboulle, Marc. Entropy-like proximal methods in convex programming. *Mathematics of Operations Research*, **19**(4)(1994) 790-814.

[11] Izuchukwu, Chinedu; Shehu, Yekini; Yao, Jen-Chih. Convergence results for proximal point algorithm with inertial and correction terms. *Appl. Anal.* **104** (2025), no. 9, 1732–1752.

[12] Liang, Jingwei; Luo, Tao; Schönlieb, Carola-Bibiane. Improving "fast iterative shrinkage-thresholding algorithm": Faster, smarter, and greedier. *SIAM Journal on Scientific Computing*, (2022),**44**(3), A1069-A1091.

[13] Moudafi, A.; Oliny, M. Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.* **155** (2003), no. 2, 447–454.

[14] Parikh, Neal; Boyd, Stephen; Proximal algorithms. *Foundations and trends in Optimization*, **1**(3), 127-239.

[15] Tibshirani, Robert. Regression shrinkage and selection via the lasso: a retrospective. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **73** (2011), no. 3, 273–282.

[16] Thung, Kim-Han; Raveendran, Paramesran. A survey of image quality measures. In *2009 international conference for technical postgraduates (TECHPOS).* IEEE, (2009). p. 1-4.

[17] Wang, Z; Bovik, A.C.; Sheikh, H.R. Simoncelli. Image quality assessment: from error visibility to structural similarity. *IEEE Transactions on Image Processing*, (2004)**13**(4), 600-612.

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