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# Approximate Solutions for Multi-Objective Optimization Problems via Scalarizing and Nonscalarizing Methods [1]

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ABSTRACT. In this paper, max-ordering and weighted compromising methods are employed to investigate approximate Pareto solutions for a class of multi-objective optimization problems with an infinite number of constraints. Approximate optimality conditions for $\varepsilon$-quasi Pareto solutions and almost $\varepsilon$-quasi Pareto solutions of the considered problems are established. The results are derived using a new notion of $\varepsilon$-quasi subdifferentials for locally Lipschitz functions and $\varepsilon$-quasi normal sets. Approximate duality theorems are also introduced. In particular, the relationships between the original multi-objective optimization problem and its dual are analyzed via a corresponding pair of primal-dual scalar problems. Several examples are provided to illustrate the proposed notions and to demonstrate the applicability of the obtained results.

## 1. INTRODUCTION

In optimization theory, necessary and sufficient optimality conditions play a crucial role, as they provide a foundation for developing optimization algorithms. Furthermore, solving an optimization problem using numerical methods often involves obtaining approximate solutions. Therefore, results obtained for exact solutions of optimization problems are frequently extended to their approximate counterparts.

The study of approximate solutions for multi-objective optimization problems has attracted considerable attention from researchers over the years (e.g., [3], [18], [24], [11], [21], [17], [30], [19], [22], [8], [25]). In this article, we focus on approximate solutions for multi-objective optimization problems with an infinite number of constraints. Building on the scalarizing and nonscalarizing methods presented in Mathias Ehrgott's book [35] for deriving exact solutions of multi-objective optimization problems, we develop approaches to formulate approximate optimality conditions for approximate Pareto solutions.

Numerous researchers have provided valuable insights into approximate solutions and numerical techniques for solving optimization problems (e.g., [9], [10], [11], [12], [13],[14], [15]). When studying approximate solutions, most works tend to focus on those lying within the feasible set of the optimization problems (see, e.g., [2], [18], [11], [17], [30], [19], [22], [8], [20], [25], [1], [26], [27], [29]). However, approximate solutions outside the feasible set also warrant attention, as they align well with numerical methods.

Therefore, in addition to type-$\varepsilon$ solutions or $\varepsilon$-quasi solutions, almost $\varepsilon$-solutions or almost $\varepsilon$-quasi solutions should also be considered. These concepts were initially proposed for scalar optimization problems by P. Loridan [4] and later extended to multi-objective optimization problems by J.C. Liu [3]. Several authors have expressed interest in further investigating these solution types (e.g., [3], [4], [23], [24], [21], [28]).

[1]Dedicated to Professors **Hoang Xuan Phu** and **Suthep Suantai** on the occasion of their milestone anniversaries

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Recently, a new notion of $\varepsilon$-quasi subdifferentials for locally Lipschitz functions was proposed in [27] and applied to establish approximate optimality conditions for certain classes of nonconvex optimization problems. This development also facilitates the representation of approximate optimality conditions for a class of nonsmooth optimization problems in several equivalent forms [28].

The aim of this study is to apply this concept to analyze approximate solutions for a class of multi-objective optimization problems. The research is divided into two parts: the first focuses on approximate optimality conditions, while the second investigates approximate duality theorems via a pair of corresponding primal-dual scalar problems.

In the first part, using both scalar and nonscalar methods, and leveraging the properties of the aforementioned $\varepsilon$-quasi subdifferentials for Lipschitz functions, we establish necessary and sufficient approximate optimality conditions for a class of multi-objective optimization problems. In addition to the commonly used scalarization method in multi-objective optimization, we also consider Tchebychev's nonscalarization method. Notably, the Tchebychev scalarization along rays, developed by Dutta and Kaya [32], is introduced in [31].

In the second part, we examine a Wolfe-type dual problem and explore the relationships between the primal and dual problems through approximate duality theorems. For a multi-objective problem denoted by (MP), we define its corresponding Wolfe-type dual problem as (MD). By applying Tchebychev's nonscalarization method to (MP), we derive a reformulated problem, denoted by (WTP), and its associated dual problem, denoted by (WTD). The relationships between (MP) and (MD) are summarized in the following diagram:

$$\begin{array}{ccc} (\mathrm{MP}) & \longrightarrow & (\mathrm{MD}) \\ \downarrow & & \uparrow \\ (\mathrm{WTP}) & \longrightarrow & (\mathrm{WTD}) \end{array}$$

We also note that this diagram was proposed in [21], where the relationships between a multi-objective optimization problem and its dual were investigated via a pair of corresponding primal-dual scalar problems based on the $\varepsilon$-constraints method. In our research, the relationships between problems (MP) and (MD) are examined through the diagram above using the max-ordering method. These scalar methods for multi-objective optimization problems can be found in the following two books: [34], [35]. It is worth noting that the relationship between a multi-objective optimization problem and its dual can also be studied via direct duality schemes. Here, by utilizing the duality schemes of the corresponding scalarized problems, we further establish the relationships between the multi-objective optimization problem and its dual. This approach leverages the rich theoretical foundation of scalar optimization.

The paper is organized as follows. The next section is devoted to fundamental notions and preliminary results. In particular, we recall results concerning the $\varepsilon$-quasi subdifferential of locally Lipschitz functions, as well as the concept of $\varepsilon$-quasi normal sets. Approximate optimality conditions for $\varepsilon$-quasi solutions of optimization problems with infinitely many constraints are also presented. The main results are provided in Sections 3 and 4. In Section 3, we focus on specific types of approximate Pareto solutions for multi-objective optimization problems, namely, almost weakly $\varepsilon$-Pareto solutions and almost $\varepsilon$-Pareto solutions. To derive these results, we employ two approaches: the max-ordering method and the weighted compromising method. These methods allow us to study approximate Pareto solutions of a multi-objective problem via approximate solutions of related scalar

optimization problems. In the final section, we investigate the relationships between almost approximate Pareto solutions of the primal multi-objective problem and its corresponding dual problem.

## 2. NOTATIONS AND BASIC RESULTS

Let $f$ be a function defined on $\mathbb{R}^n$. The function $f$ is said to be *locally Lipschitz* at a point $z$ if there exists a neighborhood $U(z) \subset \mathbb{R}^n$ and a constant $K := K(z) > 0$ such that

$$|f(x) - f(y)| \leq K\|x - y\|, \quad \forall x, y \in U(z).$$

If the inequality above is satisfied for all $x, y \in \mathbb{R}^n$, then $f$ is called *Lipschitz* on $\mathbb{R}^n$.

Let $\Omega$ be an arbitrary index set (possibly infinite). For each $\nu \in \Omega$, let $g_\nu \colon \mathbb{R}^n \to \mathbb{R}$. We say that $(g_\nu)_{\nu\in\Omega}$ is *locally Lipschitz with respect to $x$ uniformly in $\nu$* if there exists a neighborhood $U(x) \subset \mathbb{R}^n$ and a constant $K > 0$ such that

$$|g_\nu(y) - g_\nu(z)| \leq K\|y - z\|, \quad \forall y, z \in U(x) \text{ and } \forall \nu \in \Omega.$$

For a function $f$ defined on $\mathbb{R}^n$, the *domain* and the *epigraph* of $f$ are denoted and defined by

$$\mathrm{dom} f := \{x \in \mathbb{R}^n \mid |f(x)| < +\infty\}, \quad \mathrm{epi} f := \{(x, \alpha) \in \mathrm{dom} f \times \mathbb{R} \mid f(x) \leq \alpha\},$$

respectively.

The *Clarke directional derivative* of $f$ at $z \in \mathrm{dom} f$ in the direction $d \in \mathbb{R}^n$ and the *Clarke subdifferential* of $f$ at $z \in \mathrm{dom} f$ are denoted and defined, respectively, by

$$f^c(z; d) := \limsup_{\substack{x \to z \\ t \downarrow 0}} \frac{f(x + td) - f(x)}{t}, \quad \partial^c f(z) := \{u \in \mathbb{R}^n \mid \langle u, d\rangle \leq f^c(z; d),\ \forall d \in \mathbb{R}^n\}.$$

We also use the notation $f'(z; d)$ to denote the *one-sided directional derivative* of $f$ at $z \in \mathrm{dom} f$ in the direction $d$, i.e.,

$$f'(z; d) := \lim_{t \downarrow 0} \frac{f(z + td) - f(z)}{t}.$$

The function $f$ is said to be *regular* at $z$ if $f'(z; d) = f^c(z; d)$ for all $d \in \mathbb{R}^n$ (see [33]).

In this article, we also refer to *generalized convex functions*. A locally Lipschitz function $f \colon \mathbb{R}^n \to \mathbb{R}$ is said to be

(i) *Quasiconvex* if

$$f(y) \leq f(x) \Rightarrow f^c(x; y - x) \leq 0 \quad \text{for all } x, y \in \mathbb{R}^n.$$

(ii) *Pseudoconvex* if

$$f^c(x; y - x) \geq 0 \Rightarrow f(y) \geq f(x) \quad \text{for all } x, y \in \mathbb{R}^n.$$

For a closed subset $A \subset \mathbb{R}^n$, the *normal cone* of $A$ at a point $z \in A$ is given by

$$N(A, z) := \{u \in \mathbb{R}^n \mid \langle u, v\rangle \leq 0,\ \forall v \in T(A, z)\},$$

where $T(A, z) := \{v \in \mathbb{R}^n \mid d_A^c(z; v) = 0\}$ is the *tangent cone* to $A$ at $z$, and $d_A$ is the *distance function* from $z$ to $A$ [38, p. 10].

When $A$ is a convex set, the normal cone coincides with the one in convex analysis, i.e.,

$$N(A, z) := \{u \in \mathbb{R}^n \mid \langle u, x - z\rangle \leq 0,\ \forall x \in A\}.$$

Obviously, $N(A, z) = \partial\delta_A(z)$.

We recall the notion of the contingent cone. For a subset $A \subset \mathbb{R}^n$, let $z \in A$. A vector $v \in \mathbb{R}^n$ is called a *tangent* to $A$ at $z$ if for any $\varepsilon > 0$, there exist $t \in (0, \varepsilon)$ and a point $w \in v + \varepsilon B$ such that $z + tw \in A$. The collection of all such vectors $v$ is called the *contingent*

*cone* of $A$ at $z$, and is denoted by $K(A,z)$. It is clear that $T(A,z) \subset K(A,z)$. Note that $K(A,z)$ may not be convex. The set $A$ is said to be *regular* at $z$ if $T(A,z) = K(A,z)$ (see [33, p. 55]).

We also use the notation $K'$ to denote the *polar cone* of a nonempty set $K \subset \mathbb{R}^n$, i.e.,

$$K' := \{y \in \mathbb{R}^n \mid \langle y, x\rangle \leq 0,\ \forall x \in K\}.$$

Hence, $T(A,z) = [N(A,z)]'$.

For convex subsets $A$ and $B$ of $\mathbb{R}^n$, we have

$$N(A,z) + N(B,z) \subset N(A \cap B, z) \quad \text{for all } z \in A \cap B.$$

It is worth noting that if $A$ and $B$ are nonempty convex subsets of $\mathbb{R}^n$, and either $\operatorname{int} A \cap B \neq \emptyset$ or $\operatorname{int} B \cap A \neq \emptyset$, then for any $z \in A \cap B$, one has

$$N(A \cap B, z) = N(A,z) + N(B,z)$$

(see [37, Corollary 2.15]). In this case, we also have

$$T(A \cap B, z) = T(A,z) \cap T(B,z).$$

The following definitions are due to Hiriart-Urruty [6].

**Definition 2.1.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The function $f$ is called* $\varepsilon$-pseudoconvex *at $z$ if*

$$f^c(z;d) + \sqrt{\varepsilon}\|d\| \geq 0 \Rightarrow f(z+d) + \sqrt{\varepsilon}\|d\| \geq f(z), \quad \forall d \in \mathbb{R}^n.$$

*The function $f$ is called $\varepsilon$-pseudoconvex if it is $\varepsilon$-pseudoconvex at every $z \in \mathbb{R}^n$.*

It is clear that when $\varepsilon = 0$, an $\varepsilon$-pseudoconvex function reduces to a pseudoconvex function.

**Definition 2.2** ([27, Definition 3.1])**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at $z$. Given $\varepsilon \geq 0$, a vector $u \in \mathbb{R}^n$ is said to be an* $\varepsilon$-quasi subgradient *of $f$ at $z$ if*

$$f^c(z;d) + \sqrt{\varepsilon}\|d\| \geq \langle u, d\rangle, \quad \forall d \in \mathbb{R}^n.$$

*The set of all such vectors $u$ is called the* $\varepsilon$-quasi subdifferential *of $f$ at $z$, denoted by $\partial^c_\varepsilon f(z)$, i.e.,*

$$\partial^c_\varepsilon f(z) := \{u \in \mathbb{R}^n \mid f^c(z;d) + \sqrt{\varepsilon}\|d\| \geq \langle u, d\rangle,\ \forall d \in \mathbb{R}^n\}.$$

**Theorem 2.1** ([27, Theorem 3.2])**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at $z$, and let $\varepsilon \geq 0$. Then,*

$$\partial^c_\varepsilon(\lambda f)(z) = \lambda\, \partial^c_{\frac{\varepsilon}{\lambda^2}} f(z), \quad \forall \lambda \neq 0.$$

**Theorem 2.2.** *[*[27, Theorem 3.3]*] Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at $z$. Then,*

$$\partial^c_\varepsilon f(z) = \partial^c f(z) + \sqrt{\varepsilon}B,$$

*where $B$ is the closed unit ball in $\mathbb{R}^n$.*

**Theorem 2.3** ([27, Theorem 3.4])**.** *Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions at $z$. Given $\varepsilon \geq 0$, one has*

$$\partial^c_\varepsilon(f+g)(z) \subseteq \partial^c_{\varepsilon_1} f(z) + \partial^c_{\varepsilon_2} g(z),$$

*for any $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$. If $f$ and $g$ are regular at $z$, then the equality holds.*

**Definition 2.3** ([27, Definition 3.2])**.** *Let $K \subset \mathbb{R}^n$ be a nonempty set and $z \in K$. For $\varepsilon \geq 0$, the set of all* $\varepsilon$-normal vectors *of $K$ at $z$ is defined by*

$$N^Q_\varepsilon(K,z) := \{u \in \mathbb{R}^n \mid \langle u, d\rangle \leq \sqrt{\varepsilon}\|d\|,\ \forall d \in T(K,z)\}.$$

When $\varepsilon = 0$, $N_\varepsilon^Q(K,z)$ reduces to the normal cone of $K$ in the sense of Clarke. Furthermore, if $K$ is a convex set, then we recover the convex normal cone $N(K,z)$ in the sense of convex analysis.

**Proposition 2.1** ([27, Proposition 3.7]). *Let $K \subset \mathbb{R}^n$ be a nonempty set and $z \in K$. For $\varepsilon \geq 0$, one has*

$$N(K,z) + \sqrt{\varepsilon}B \subseteq N_\varepsilon^Q(K,z).$$

*The equality holds if $K$ is convex.*

The function $f$ is called *regular* at $z$ if $\operatorname{epi} f$ is regular at $(z, f(z))$. We note that when $f$ is locally Lipschitz at $z$, the set $\partial^c f(z)$ coincides with the Clarke subdifferential of $f$ at $z$.

**Lemma 2.1** ([27, Proposition 3.10]). *Let $g_i \colon \mathbb{R}^n \to \mathbb{R}$, for $i \in I := \{1, 2, \ldots, m\}$, be locally Lipschitz functions at $z$. Let*

$$g(z) := \max_{i \in I} g_i(z), \quad I(z) := \{i \in I \mid g(z) = g_i(z)\}.$$

*Given $\varepsilon > 0$, one has*

$$\partial_\varepsilon^c g(z) \subseteq \operatorname{co}\left\{ \bigcup_{i \in I(z)} \partial_{\varepsilon_i}^c g_i(z) \right\},$$

*where $\sqrt{\varepsilon}$ is a convex combination of $\sqrt{\varepsilon_i}$, $i \in I(z)$. The equality holds if the functions $g_i$, $i \in I$, are regular at $z$.*

We are now in a position to study approximate optimality conditions for multi-objective optimization problems with an infinite number of constraints via nonscalar methods. For an index set $\Omega$, define

$$\mathbb{R}^{(\Omega)} := \{\lambda = (\lambda_\nu)_{\nu \in \Omega} \mid \lambda_\nu = 0 \text{ for all but finitely many } \nu \in \Omega\}.$$

This linear space was first introduced in [7] and is commonly used in semi-infinite programming [36]. The nonnegative cone of $\mathbb{R}^{(\Omega)}$ is denoted by

$$\mathbb{R}_+^{(\Omega)} := \left\{ \lambda = (\lambda_\nu)_{\nu \in \Omega} \in \mathbb{R}^{(\Omega)} \mid \lambda_\nu \geq 0,\ \forall \nu \in \Omega \right\}.$$

For $\lambda \in \mathbb{R}_+^{(\Omega)}$, the *supporting set* of $\lambda$, denoted by $\Omega(\lambda)$, is defined by

$$\Omega(\lambda) := \{\nu \in \Omega \mid \lambda_\nu \neq 0\},$$

which is a finite subset of $\Omega$. It is clear that the function

$$\lambda \mapsto \sum_{\nu \in \Omega(\lambda)} |\lambda_\nu|$$

is a norm. We define

$$\|\lambda\|_1 := \sum_{\nu \in \Omega(\lambda)} |\lambda_\nu|.$$

We use the notation $\langle \cdot, \cdot \rangle$ to denote the canonical bilinear form of the duality pairing on $\mathbb{R}^{(\Omega)} \times \mathbb{R}^\Omega$. For $\lambda \in \mathbb{R}^{(\Omega)}$ and $z = \{z_\nu\}_{\nu \in \Omega} \subset \mathbb{R}^\Omega$, we define

$$\langle \lambda, z \rangle := \sum_{\nu \in \Omega} \lambda_\nu z_\nu = \begin{cases} \sum_{\nu \in \Omega(\lambda)} \lambda_\nu z_\nu, & \text{if } \Omega(\lambda) \neq \emptyset, \\ 0, & \text{if } \Omega(\lambda) = \emptyset. \end{cases}$$

The following notations are also used in this paper:

$$\begin{aligned} \mathbb{R}_+^m &:= \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geq 0,\ i = 1, \ldots, m\}, \\ \mathbb{R}_{++}^m &:= \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i > 0,\ i = 1, \ldots, m\}. \end{aligned}$$

We now recall some concepts of approximate solutions for a scalar optimization problem. Consider the following scalar optimization problem:

$$\begin{aligned} \text{(P)} \quad & \text{Minimize} && f(x) \\ & \text{subject to} && g_\nu(x) \le 0, \quad \nu \in \Omega, \\ & && x \in S, \end{aligned}$$

where $\Omega$ is an index set (possibly infinite), $S$ is a nonempty and closed subset of $\mathbb{R}^n$, $f\colon \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, and the constraint functions $g_\nu\colon \mathbb{R}^n \to \mathbb{R}$, $\nu \in \Omega$, are locally Lipschitz with respect to $x$, uniformly in $\nu$.

Let us denote by

$$F := \{x \in S \mid g_\nu(x) \le 0,\ \forall \nu \in \Omega\}$$

the feasible set of (P), and by

$$A := \{x \in \mathbb{R}^n \mid g_\nu(x) \le 0,\ \forall \nu \in \Omega\},$$

the set determined by the constraint functions of (P) alone. Clearly, $F = A \cap S$.

In most cases, approximate solutions of an optimization problem are required to lie within its feasible set. However, this requirement can be too strict for certain numerical methods. To facilitate the study of approximate solutions to (P), we consider an expansion of the feasible set.

Given $\alpha > 0$, a relaxation of the feasible set $F$, with tolerance $\alpha$, is denoted and defined by

$$F_\alpha := \{x \in S \mid g_\nu(x) \le \sqrt{\alpha},\ \forall \nu \in \Omega\}. \tag{2.1}$$

In this research, the value of the tolerance $\alpha > 0$ is selected appropriately based on the specific problem under consideration.

It is clear that $F \subseteq F_\alpha$. Following [4], we now recall the following concepts of approximate solutions.

**Definition 2.4.** *Let $\alpha > 0$. A point $z \in F_\alpha$ is said to be*

(i) *an* almost $\alpha$-solution *if $f(z) \le f(x) + \alpha$ for all $x \in F$;*
(ii) *an* almost quasi $\alpha$-solution *if $f(z) \le f(x) + \sqrt{\alpha}\|x - z\|$ for all $x \in F$;*
(iii) *an* almost regular $\alpha$-solution *if $z$ is both an $\alpha$-solution and a quasi $\alpha$-solution.*

*When $z \in F$, we recover the concepts of $\alpha$-solution, quasi $\alpha$-solution, and regular $\alpha$-solution, respectively.*

For problem (P), define

$$I(z) := \{\nu \in \Omega \mid g_\nu(z) = 0\}.$$

We will need the following constraint qualification condition:

$$(\mathcal{H}): \quad \exists d \in \mathbb{R}^n \text{ such that } g_\nu^c(z; d) < 0 \quad \text{for all } \nu \in I(z).$$

We now recall some results on approximate solutions for (P) with a tolerance $\varepsilon > 0$, as introduced in [23]. Note that approximate optimality conditions for (P) can be presented in various forms (see [28, Corollary 3.2]). One of them is given below.

For the compact index set $\Omega$, and inspired by [28, Proposition 3.2], we propose the following theorem with the proof omitted.

**Theorem 2.4.** *For problem* (P), *let $z \in A$. Suppose that condition $(\mathcal{H})$ holds at $z$. If $z$ is an $\varepsilon$-quasi solution of* (P), *then there exist constants $\bar{\varepsilon} \ge 0$, $\hat{\varepsilon} \ge 0$, and scalars $\bar{\lambda}_\nu \ge 0$, $\varepsilon_\nu \ge 0$ for all $\nu \in \Omega$, such that*

$$0 \in \partial^c_{\bar{\varepsilon}} f(z) + \sum_{\nu \in \Omega(\bar{\lambda})} \bar{\lambda}_\nu\, \partial^c_{\varepsilon_\nu / \bar{\lambda}_\nu^2} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z), \tag{2.2}$$

*with*

$$g_\nu(z) = 0, \quad \forall \nu \in \Omega(\bar{\lambda}), \quad \textit{and} \quad \sqrt{\varepsilon} = \sqrt{\bar{\varepsilon}} + \sum_{\nu \in \Omega(\bar{\lambda})} \bar{\lambda}_\nu \sqrt{\varepsilon_\nu} + \sqrt{\hat{\varepsilon}}. \tag{2.3}$$

Also motivated by [28, Proposition 3.3], we propose the following theorem without proof.

**Theorem 2.5.** *For problem* (P)*, suppose that the objective function $f$ is $\varepsilon$-pseudoconvex and the constraint functions $g_\nu$, $\nu \in \Omega$, are quasiconvex. If conditions* (2.2)–(2.3) *are satisfied for some $\bar{\varepsilon} \geq 0$, $\hat{\varepsilon} \geq 0$, and $\lambda_\nu \geq 0$, $\varepsilon_\nu \geq 0$, $\nu \in \Omega$, then the point $z \in F$ is a quasi $\varepsilon$-solution of* (P)*, provided that*

$$g_\nu(x) \leq g_\nu(z), \quad \forall \nu \in \Omega, \ \forall x \in A.$$

Given $\varepsilon > 0$, we now discuss almost approximate solutions for problem (P) using the relaxed feasible set $F_\varepsilon$. In the following theorem, the existence of almost approximate solutions for (P) is established without requiring a constraint qualification condition.

**Theorem 2.6** ([28, Theorem 3.2])**.** *Let $\varepsilon > 0$. Then, there exist $z_\varepsilon \in F_\varepsilon$, which is an almost regular $\varepsilon$-solution for* (P)*, $\mu \in \mathbb{R}_+^{(\Omega)}$, and nonnegative scalars $\hat{\varepsilon}, \bar{\varepsilon}, \varepsilon_\nu \geq 0$, for all $\nu \in \Omega$, such that*

$$\begin{cases} 0 \in \partial^c_{\bar{\varepsilon}} f(z_\varepsilon) + \displaystyle\sum_{\nu \in \Omega(\mu)} \mu_\nu \, \partial^c_{\varepsilon_\nu} g_\nu(z_\varepsilon) + N^Q_{\hat{\varepsilon}}(C, z_\varepsilon), \\ \qquad\qquad g_\nu(z_\varepsilon) > 0, \quad \forall \nu \in \Omega(\mu), \\ \sqrt{\varepsilon} = \sqrt{\bar{\varepsilon}} + \displaystyle\sum_{\nu \in \Omega(\mu)} \mu_\nu \sqrt{\varepsilon_\nu} + \sqrt{\hat{\varepsilon}}. \end{cases}$$

## 3. Approximate Pareto Solutions for Multi-Objective Problems

In this section, we are concerned with approximate optimality conditions for a class of multi-objective optimization problems with an infinite number of constraints. The scalarizing and nonscalarizing methods used here are based on those introduced in the book by M. Ehrgott [35].

Consider the following problem:

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) := \big(f_1(x), f_2(x), \ldots, f_m(x)\big) \\ & \text{subject to} & g_\nu(x) \leq 0, \quad \nu \in \Omega, \\ & & x \in S, \end{array}$$

where $\Omega$, $S$, and the constraint functions $g_\nu$, $\nu \in \Omega$, are defined as before, and the objective functions $f_i$, $i \in M := \{1, 2, \ldots, m\}$, are locally Lipschitz functions defined on $\mathbb{R}^n$.

We use the notation $F$ to denote the feasible set of (MP), and $A$ to denote the set determined solely by the constraint functions of (MP). Thus, we have $F = A \cap S$.

To handle almost approximate solutions of a multi-objective optimization problem, we consider a relaxation of its feasible set. Let $\varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \in \mathbb{R}^m_{++}$ be a vector of tolerances. We introduce a scalar quantity $\theta$ that depends on $\varepsilon$, defined by

$$\theta := \min_{1 \leq i \leq m} \varepsilon_i.$$

Similar to (2.1), the relaxation of the feasible set of (MP) is denoted by $F_\theta$ in the following theorem.

**Definition 3.5.** [24, Definition 2.9] *A point $z \in F_\theta$ is said to be:*

(i) *an* almost weakly $\varepsilon$-Pareto solution *of* (MP) *if there exists no $x \in F$ such that*

$$f_i(x) < f_i(z) - \varepsilon_i, \quad \forall i \in M;$$

(ii) *an* almost weakly quasi $\varepsilon$-Pareto solution *of* (MP) *if there exists no* $x \in F$ *such that*

$$f_i(x) + \sqrt{\varepsilon_i}\|x - z\| < f_i(z), \quad \forall i \in M;$$

(iii) *an* almost weakly regular $\varepsilon$-Pareto solution *of* (MP) *if it is both an almost weakly quasi $\varepsilon$-Pareto solution and an almost weakly $\varepsilon$-Pareto solution.*

If $z \in F$, then the term "almost" can be omitted. In that case, we obtain the concepts of weakly $\varepsilon$-Pareto solution, weakly quasi $\varepsilon$-Pareto solution, and weakly regular $\varepsilon$-Pareto solution of (MP), respectively.

**Definition 3.6.** *A point $z \in F_\theta$ is said to be:*

(i) *an* almost $\varepsilon$-Pareto solution *of* (MP) *if there exists no* $x \in F$ *such that*

$$\begin{cases} f_i(x) \leq f_i(z) - \varepsilon_i, \quad \forall i \in M, \\ \exists j \in M : f_j(x) < f_j(z) - \varepsilon_j; \end{cases}$$

(ii) *an* almost quasi $\varepsilon$-Pareto solution *of* (MP) *if there exists no* $x \in F$ *such that*

$$\begin{cases} f_i(x) + \sqrt{\varepsilon_i}\|x - z\| \leq f_i(z), \quad \forall i \in M, \\ \exists j \in M : f_j(x) + \sqrt{\varepsilon_j}\|x - z\| < f_j(z); \end{cases}$$

(iii) *an* almost regular $\varepsilon$-Pareto solution *of* (MP) *if it is both an almost $\varepsilon$-Pareto solution and an almost quasi $\varepsilon$-Pareto solution.*

If $z \in F$, then we recover the standard notions of $\varepsilon$-Pareto solution, quasi $\varepsilon$-Pareto solution, and regular $\varepsilon$-Pareto solution of (MP), respectively.

**Example 1**

$$\begin{array}{ll} \text{Minimize} & \big(-|x - \frac{\pi}{2}|, \sin x\big) \\ \text{subject to} & \frac{5\pi}{12} \leq x \leq \frac{5\pi}{6}. \end{array} \tag{3.4}$$

Let $F := \{x \in \mathbb{R} \mid \frac{5\pi}{12} \leq x \leq \frac{5\pi}{6}\}$. We get $\frac{\pi}{3}, \pi \notin F$. We now check $z = \frac{\pi}{3}$ is an almost quasi $\varepsilon$-Pareto solution and $z = \pi$ is an almost $\varepsilon$-Pareto solution of the bi-objective problem for $\varepsilon = (\frac{1}{4}, \frac{1}{4})$. Let

$$h_1(x) = -|x - \frac{\pi}{2}|, h_2(x) = \sin x.$$

i) *$z = \pi$ is an almost $\varepsilon$-Pareto solution of the bi-objective problem*: Note that, for all $x \in F$, $h_1(x) \geq h_1(\pi)$ and $h_2(x) \geq h_2(\pi)$ for all $x \in F$. Hence, $h_1(x) \geq h_1(\pi) - \frac{1}{4}$ and $h_2(x) \geq h_2(\pi) - \frac{1}{4}$. This implies, by Definition 3.6 (i), that $z = \pi$ is an almost $\varepsilon$-solution of the problem.

ii) *$z = \frac{\pi}{3}$ is an almost quasi $\varepsilon$-Pareto solution of the bi-objective problem*: Since the inequality $-\frac{\pi}{6} \leq -|x - \frac{\pi}{2}| + \frac{1}{2}|\frac{\pi}{3} - x|$ holds with $x \in [\frac{\pi}{3}, \pi]$, we get

$$h_1(\pi/3) \leq h_1(x) + \frac{1}{2}|\frac{\pi}{3} - x|, \forall x \in F. \tag{3.5}$$

Moreover, since $|\sin\frac{\pi}{3} - \sin x| \leq \frac{1}{2}|\frac{\pi}{3} - x|$ for all $x \in [\pi/3, \pi]$, it follows that $\sin\frac{\pi}{3} \leq \sin x + \frac{1}{2}|\frac{\pi}{3} - x|$ for all for all $x \in [\pi/3, \pi]$. Hence,

$$h_2(\pi/3) \leq h_2(x) + \frac{1}{2}|\frac{\pi}{3} - x|, \forall x \in F. \tag{3.6}$$

It follows from (3.5) and (3.6) that $z = \frac{\pi}{3}$ is is an almost quasi $\varepsilon$-Pareto solution of the bi-objective problem in the sense of Definition 3.6 (ii).

3.1. **Max-Ordering Method.** We begin with a method for multi-objective problems, known as the *Max-ordering method*. This method is also known as a nonscalarizing method [35]. Associated with problem (MP), define the scalar function:

$$G(x) := \max\{f_1(x), f_2(x), \ldots, f_m(x)\}.$$

We then consider the following scalar optimization problem:

$$\begin{array}{lll} \text{(MOP)} & \text{Minimize} & G(x) \\ & \text{subject to} & g_\nu(x) \leq 0, \quad \nu \in \Omega, \\ & & x \in S, \end{array}$$

where $S$, $g_\nu$, and $\nu \in \Omega$ are defined as before. In this formulation, (MOP) becomes a single-objective optimization problem. We continue to use the notation $F$ to denote the feasible set of (MOP).

Note that when all $\varepsilon_i$, $i \in M$, are sufficiently small, and we set $\theta := \min_{i\in M} \varepsilon_i$, then it follows that

$$\sqrt{\theta} = \min_{i\in M} \sqrt{\varepsilon_i}.$$

Inspired by [35, Definition 5.8] and [35, Proposition 5.9], we now propose the following definition and associated lemma for problem (MOP).

**Definition 3.7.** *A point $z \in F_\theta$ is called an* almost quasi $\theta$-solution *for* (MOP) *in the Max-ordering sense if there exists no $x \in F$ such that*

$$\max_{k\in M} \left\{ f_k(x) + \sqrt{\theta}\|x - z\| \right\} < \max_{k\in M} f_k(z).$$

**Example 2:**

Let us reconsider the problem given by (3.4). For $h_1(x) = -|x - \pi/2|, h_2(x) = \sin x$. It is clear that $h_1(x) \leq h_2(x)$ for all $x \in F$. Hence,

$$\max\{h_1(x), h_2(x)\} = h_2(x) = \sin x, \forall x \in F.$$

According to (3.6), with $\sqrt{\theta} = 1/2$, there is no $x \in F$ such that

$$h_2(x) + \frac{1}{2}|\frac{\pi}{3} - x| < h_2(\pi/3).$$

Since $z = \frac{\pi}{3} \notin F$, the point $z$ is an almost quasi $\theta$-solution for the problem (3.4) in the Max-ordering sense.

**Lemma 3.2.** *A point $z \in F_\theta$ is an almost quasi $\theta$-solution for the problem* MOP *in max-ordering type if and only if it is an almost weakly quasi $\varepsilon$-Pareto solution for* (MP).

*Proof.* Let $z \in F_\theta$ be an almost max-ordering quasi $\theta$-solution for (MOP). Suppose to the contrary that $z$ is not an almost weakly quasi $\varepsilon$-Pareto solution for (MP). Then, there exists $x' \in F$ such that

$$f_i(x') + \sqrt{\varepsilon_i}\|x' - z\| < f_i(z), \quad \forall i \in M.$$

Hence,

$$f_i(x') + \sqrt{\theta}\|x' - z\| < f_i(z), \quad \forall i \in M.$$

Thus,

$$\max_{i\in M} f_i(x') + \sqrt{\theta}\|x' - z\| < \max_{i\in M} f_i(z),$$

a contradiction to Definition 3.7. Hence, $z$ is an almost weakly quasi $\varepsilon$-Pareto solution for (MP).

Conversely, let $z \in F_\theta$ be an almost weakly quasi $\varepsilon$-Pareto solution for (MP). Suppose that $z$ is not an almost max-ordering quasi $\theta$-solution for (MOP). Then there exists $x' \in F$ such that

$$\max_{i\in M} f_i(x') + \sqrt{\theta}\|x' - z\| < \max_{i\in M} f_i(z),$$

i.e., there exists $j \in M$ such that

$$f_j(x') + \sqrt{\varepsilon_j}\|x' - z\| < f_j(z).$$

This implies that $z$ is not an almost weakly quasi $\varepsilon$-Pareto solution for (MP), a contradiction. □

**Remark 3.1.** In Definition 3.7 and in Lemma 3.2, if we take $z \in F$, the word "almost" will be removed.

In the following theorem, we present an approximate necessary optimality condition for (MP) via the max-ordering method. For $G(x) = \max_{i\in M}\{f_i(x)\}$ as above, set $M_G(z) := \{i \in M \mid G(z) = f_i(z)\}$.

**Theorem 3.7.** *Let $z \in F$ be a weakly quasi $\epsilon$-Pareto solution of* (MP). *If the condition $(\mathcal{H})$ holds for $z$, then there exist $\varepsilon^* \geq 0$, $\hat{\varepsilon} \geq 0$, $\dot{\varepsilon}_\nu, \lambda_\nu \geq 0$ for $\nu \in \Omega$, and $\zeta_i \geq 0$, $i = 1, \ldots, m$, such that*

$$0 \in \sum_{i\in M_G(z)} \zeta_i \partial^c_{\bar{\varepsilon}_i} f_i(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu/\lambda_\nu^2} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z)$$

*with*

$$g_\nu(z) = 0,\ \nu \in \Omega(\lambda), \qquad \sum_{i\in M_G(z)} \zeta_i\sqrt{\bar{\varepsilon}_i} + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu\sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta}.$$

*Proof.* Let $z \in F$ be a weakly quasi $\varepsilon$-quasi solution for $(\mathrm{MP})$. Due to Lemma 3.2 and Remark 3.1, the point $z$ is a max-ordering quasi $\theta$-solution for (MOP). Hence, for $G(x) = \max_{i\in M}\{f_i(x)\}$, there is no $x' \in F$ such that

$$G(x') + \sqrt{\theta}\|x' - z\| < G(z).$$

This is equivalent to

$$G(z) \leq G(x) + \sqrt{\theta}\|x - z\|, \quad \forall x \in F,$$

i.e., $z$ is a $\theta$-quasi solution for the problem (MOP).

Since the condition $(\mathcal{H})$ is satisfied, due to Theorem 2.4, there exist $\varepsilon^*, \hat{\varepsilon}, \dot{\varepsilon}_\nu, \lambda_\nu \geq 0$, $\nu \in \Omega(\lambda)$, such that

$$0 \in \partial^c_{\varepsilon^*} G(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu/\lambda_\nu^2} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z), \tag{3.7}$$

with

$$g_\nu(z) = 0,\ \nu \in \Omega(\lambda), \quad \sqrt{\varepsilon^*} + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu\sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta}. \tag{3.8}$$

For the set $M_G(z)$, due to [27, Proposition 3.4], we have

$$\partial^c_{\varepsilon^*} G(z) \subseteq \mathrm{co} \bigcup_{i\in M_G(z)} \partial^c_{\bar{\varepsilon}_i} f_i(z), \tag{3.9}$$

for any $\bar{\varepsilon}_i \geq 0$, $i \in M_G(z)$, such that $\sqrt{\varepsilon^*}$ is a convex combination of $\sqrt{\bar{\varepsilon}_i}$. Let $\zeta_i \in [0, 1]$, $i \in M_G(z)$, where $\sum_{i\in M_G(z)} \zeta_i = 1$. We get $\sqrt{\varepsilon^*} = \sum_{i\in M_G(z)} \zeta_i\sqrt{\bar{\varepsilon}_i}$.

From (3.17), (3.8), and (3.9), we get

$$0 \in \sum_{i\in M_G(z)} \zeta_i \partial^c_{\bar{\varepsilon}_i} f_i(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\hat{\varepsilon}_\nu/\lambda^2_\nu} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S,z)$$

and

$$g_\nu(z) = 0,\ \nu \in \Omega(\lambda), \qquad \sum_{i\in M_G(z)} \zeta_i \sqrt{\bar{\varepsilon}_i} + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \sqrt{\hat{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta}.$$

□

We now show that there exists an almost weakly quasi $\varepsilon$-Pareto solution for (MP) via max-ordering method without constraint qualification condition.

**Theorem 3.8.** *There exist an almost weakly quasi $\varepsilon$-Pareto solution $z_\varepsilon \in F_\theta$ for* (MP)*, and $\mu \in \mathbb{R}^{(\Omega)}_+$, $\hat{\varepsilon}, \bar{\varepsilon}, \varepsilon_\nu \geq 0$, for all $\nu \in \Omega$, and $\zeta_i \in [0,1]$, $\sum_{i\in M_G(z)} \zeta_i = 1$, such that*

$$\begin{cases} 0 \in \sum\limits_{i\in M_G(z_\varepsilon)} \zeta_i \partial^c_{\bar{\varepsilon}_i} f_i(z_\varepsilon) + \sum\limits_{\nu\in\Omega(\mu)} \mu_\nu \partial^c_{\varepsilon_\nu} g_\nu(z_\varepsilon) + N^Q_{\hat{\varepsilon}}(C, z_\varepsilon), g_\nu(z_\varepsilon) > 0, \nu \in \Omega(\mu), \\ \sum\limits_{i\in M_G(z)} \zeta_i \sqrt{\bar{\varepsilon}_i} + \sum\limits_{\nu\in I(z)} \lambda_\nu \sqrt{\varepsilon_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta}. \end{cases}$$

*Proof.* For $G(x) = \max_{i\in M}\{f_i(x)\}$, let us reconsider the problem (MOP). For $\varepsilon \in \mathbb{R}^m_{++}$, $\sqrt{\theta} = \min_{i\in M} \sqrt{\varepsilon_i}$, by Theorem 2.6, there exists $z_\varepsilon \in F_\theta$, that is an almost regular $\theta$-solution for (MOP), $\mu \in \mathbb{R}^{(\Omega)}_+$ and $\hat{\varepsilon}, \varepsilon^*, \varepsilon_\nu \geq 0$, for all $\nu \in \Omega$, such that

$$\text{(3.10)} \qquad \begin{cases} 0 \in \partial^c_{\varepsilon^*} G(z_\varepsilon) + \sum\limits_{\nu\in\Omega(\mu)} \mu_\nu \partial^c_{\varepsilon_\nu} g_\nu(z_\varepsilon) + N^Q_{\hat{\varepsilon}}(C, z_\varepsilon), g_\nu(z_\varepsilon) > 0, \nu \in \Omega(\mu), \\ \sqrt{\theta} = \sqrt{\varepsilon^*} + \sum\limits_{\nu\in\Omega(\mu)} \mu_\nu \sqrt{\varepsilon_\nu} + \sqrt{\hat{\varepsilon}}. \end{cases}$$

The point $z_\varepsilon$ is an almost max-ordering quasi $\theta$-solution for (MOP). By Lemma 3.2, it is an almost weakly quasi $\varepsilon$-Pareto solution for (MP). On the other hand, for $M_G(z) = \{i \in M \mid G(z) = f_i(z)\}$, due to [27, Proposition 3.4], we have

$$\partial^c_{\varepsilon^*} G(z) \subseteq \text{co} \bigcup_{M_G(z)} \partial^c_{\bar{\varepsilon}_i} f_i(z),$$

for any $\bar{\varepsilon}_i, i \in M_G(z)$, such that $\sqrt{\varepsilon^*}$ is a convex combination of $\sqrt{\bar{\varepsilon}_i}, i \in M_G(z)$. Hence, there exist $\zeta_i \in [0,1]$, $\sum_{i\in M_G(z)} \zeta_i = 1$ such that $\sqrt{\varepsilon^*} = \sum_{i\in M_G(z)} \zeta_i \sqrt{\bar{\varepsilon}_i}$. The formula (3.10) is rewritten by

$$\begin{cases} 0 \in \sum\limits_{i\in M_G(z_\varepsilon)} \zeta_i \partial^c_{\bar{\varepsilon}_i} f_i(z_\varepsilon) + \sum\limits_{\nu\in\Omega(\mu)} \mu_\nu \partial^c_{\varepsilon_\nu} g_\nu(z_\varepsilon) + N^Q_{\hat{\varepsilon}}(C, z_\varepsilon), g_\nu(z_\varepsilon) > 0, \nu \in \Omega(\mu), \\ \sum\limits_{i\in M_G(z)} \zeta_i \sqrt{\bar{\varepsilon}_i} + \sum\limits_{\nu\in I(z)} \lambda_\nu \sqrt{\varepsilon_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta}. \end{cases}$$

□

3.2. **Weighted Compromising Method.** From problem (MP), suppose that, for each $i \in M$, there exists

$$y_i = \min_{x\in F} f_i(x),$$

where $F$ is the feasible set of (MP). The vector $y^I = (y_1, y_2, \ldots, y_m)$ is called an ideal point of problem (MP). Note that there may not exist $x_0 \in F$ such that $f_i(x_0) = y_i$ for all $i \in M$. However, this ideal point can serve as a reference point when we seek approximate

Pareto solutions of (MP). We will find $\bar{x} \in F$ to minimize the distance between the vectors $K(x) := (f_1(x), \ldots, f_m(x))$ and $y^I$. The following optimization model can be found in [35].

Let us consider the following problem:

$$\begin{array}{ll} \text{Minimize} & \|K(x) - y^I\|_p \\ \text{subject to} & x \in F, \end{array} \tag{3.11}$$

where $p$ indicates the type of metric used.

When $p = 1$, we get the Manhattan metric defined by $\|u - v\|_1 := \sum_{i=1}^m |u_i - v_i|$, where $u, v \in \mathbb{R}^m$.

When $p = \infty$, we get the Tchebyshev metric defined by $\|u - v\|_\infty := \max_i |u_i - v_i|,\ i = 1, \ldots, m$ (see [35] and [5]).

Set

$$\Sigma := \left\{ \xi \in (0,1)^m \mid \sum_{i=1}^m \xi_i = 1 \right\}.$$

Combining the above-mentioned metrics with a weighted vector $\xi \in \Sigma$, we are concerned with the following problems:

$$\text{(WMP)} \quad \begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^m \xi_i |f_i(x) - y_i| \\ \text{subject to} & x \in F, \end{array}$$

and

$$\text{(WTP)} \quad \begin{array}{ll} \text{Minimize} & \displaystyle\max_{1 \le i \le m} \xi_i |f_i(x) - y_i| \\ \text{subject to} & x \in F. \end{array}$$

3.2.1. *The Problem with the Weighted Manhattan Metric Method.* Given $\varepsilon \in \mathbb{R}^m_{++}$ and a weighted vector $\xi \in \Sigma$, for compatibility with the method used in this part, we set

$$\bar{\theta} := \left( \sum_{i=1}^m \xi_i \sqrt{\varepsilon_i} \right)^2.$$

In this situation, the relaxation set of (MP) is understood with $\bar{\theta}$.

Note that $y_i \le f_i(x)$ for all $i \in M$ and for all $x \in F$. Since $\sum_{i=1}^m \xi_i y_i$ is a constant, the problem (WMP) above is equivalent to the following problem:

$$(\mathrm{WMP}_1) \quad \begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^m \xi_i f_i(x) \\ \text{s.t.} & x \in F. \end{array}$$

In this case, we can see that the above problem is the one obtained by a scalarizing method named the weighted sum method [35, p. 113] (see also [5]).

We now study approximate solutions for $(\mathrm{WMP}_1)$.

**Lemma 3.3.** *A point $z \in F_{\bar{\theta}}$ is an almost quasi $\bar{\theta}$-solution for $(\mathrm{WMP}_1)$ if and only if $z$ is an almost weakly quasi $\varepsilon$-Pareto solution for* (MP).

*Proof.* Let $z \in F_{\bar{\theta}}$ be an almost quasi $\bar{\theta}$-solution for $(\mathrm{WMP}_1)$. Suppose to the contrary that $z$ is not an almost weakly quasi $\varepsilon$-Pareto solution for (MP). Then, there exists $x' \in F$ such that

$$f_i(x') + \sqrt{\varepsilon_i}\|x' - z\| < f_i(z), \quad \forall i \in M.$$

For any $\xi \in \Sigma$, we get

$$\xi_i\big(f_i(x') + \sqrt{\varepsilon_i}\|x' - z\|\big) < \xi_i f_i(z), \quad \forall i \in M.$$

Thus,

$$\sum_{i=1}^{m} \xi_i f_i(x') + \sqrt{\bar{\theta}}\|x' - z\| < \sum_{i=1}^{m} \xi_i f_i(z),$$

a contradiction to the fact that $z$ is an almost $\bar{\theta}$-quasi solution of $(\mathrm{WMP}_1)$.

Conversely, let $z$ be an almost weakly quasi $\varepsilon$-Pareto solution for (MP). Suppose to the contrary that $z$ is not an almost $\bar{\theta}$-solution for $(\mathrm{WMP}_1)$. Then there exists $x' \in F$ such that

$$\sum_{i=1}^{m} \xi_i\big(f_i(x') + \sqrt{\varepsilon_i}\|x' - z\|\big) < \sum_{i=1}^{m} \xi_i f_i(z).$$

This implies that there exists $j \in M$ such that the following strict inequality holds with $x'$,

$$f_j(x') + \sqrt{\varepsilon_j}\|x' - z\| < f_j(z),$$

a contradiction. □

**Remark 3.2.** In the lemma above, if we take $z \in F$, then the conclusion of the lemma has no occurrence of the word "almost".

**Theorem 3.9.** *Suppose that the condition $(\mathcal{H})$ holds for $z \in F$. If $z$ is a weakly quasi $\varepsilon$-Pareto solution for* (MP)*, then there exist $\varepsilon^* \geq 0$, $\hat{\varepsilon} \geq 0$, $\dot{\varepsilon}_\nu \geq 0$, and $\lambda_\nu \geq 0$ for all $\nu \in \Omega$ such that*

$$0 \in \sum_{i=1}^{m} \xi_i \partial^c_{\varepsilon^*} f_i(z) + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu/\lambda_\nu^2} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z)$$

*and*

$$g_\nu(z) = 0,\ \nu \in \Omega(\lambda), \quad \sqrt{\varepsilon^*} + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sum_{i=1}^{m} \xi_i \sqrt{\varepsilon_i}.$$

*Proof.* If $z$ is a weakly quasi $\varepsilon$-Pareto solution for (MP), then by Lemma 3.3, it is a $\bar{\theta}$-quasi solution for $(\mathrm{WMP}_1)$. Let $G(x) = \sum_{i=1}^{m} \xi_i f_i(x)$ with $\xi \in \Sigma$. Based on the condition $(\mathcal{H})$, and due to [28, Corollary 3.2], there exist $\varepsilon^*, \hat{\varepsilon}, \dot{\varepsilon}_\nu, \lambda_\nu \geq 0$, $\nu \in I(z)$, such that

$$0 \in \partial^c_{\varepsilon^*} G(z) + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu/\lambda_\nu^2} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z),$$

with

$$g_\nu(z) = 0,\ \nu \in \Omega(\lambda), \quad \sqrt{\varepsilon^*} + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sum_{i=1}^{m} \xi_i \sqrt{\varepsilon_i}.$$

Since

$$\begin{aligned} \partial^c_{\varepsilon^*} G(z) &\subseteq \textstyle\sum_{i=1}^{m} \xi_i \partial^c f_i(z) + \sqrt{\varepsilon^*} B \\ &= \textstyle\sum_{i=1}^{m} \xi_i \left[\partial^c f_i(z) + \sqrt{\varepsilon^*} B\right] \\ &= \textstyle\sum_{i=1}^{m} \xi_i \partial^c_{\varepsilon^*} f_i(z), \end{aligned}$$

we get

$$0 \in \sum_{i=1}^{m} \xi_i \partial^c_{\varepsilon^*} f_i(z) + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z),$$

$$g_\nu(z) = 0,\ \nu \in \Omega(\lambda), \quad \sqrt{\varepsilon^*} + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sum_{i=1}^{m} \xi_i \sqrt{\varepsilon_i}.$$

□

We now study almost approximate Pareto solutions for (MP) by using a scalarizing method under the Manhattan metric. For $\bar{\theta}$ given above, let us consider the relaxation of the feasible set of $(\mathrm{WMP}_1)$, given by $F_{\bar{\theta}}$.

**Theorem 3.10.** *There exists $z \in F_{\bar{\theta}}$ that is an almost regular $\bar{\theta}$-solution for $(\mathrm{WMP}_1)$, and $\varepsilon^*, \hat{\varepsilon}, \dot{\varepsilon}_\nu, \lambda_\nu \geq 0$, $\nu \in \Omega(\lambda)$, such that*

$$0 \in \sum_{i=1}^{m} \xi_i \partial^c_{\varepsilon^*} f_i(z) + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z), \quad g_\nu(z) > 0,\ \nu \in \Omega(\lambda), \tag{3.12}$$

*with*

$$\sqrt{\varepsilon^*} + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sum_{i=1}^{m} \xi_i \sqrt{\varepsilon_i}. \tag{3.13}$$

*Proof.* For $G(x) = \sum_{i=1}^{m} \xi_i f_i(x)$, it is clear that $G$ is a locally Lipschitz function. By [28, Theorem 3.9], there exists an almost regular $\bar{\theta}$-solution for $(\mathrm{WMP}_1)$, and $\varepsilon^*, \hat{\varepsilon}, \dot{\varepsilon}_\nu, \lambda_\nu \geq 0$, $\nu \in \Omega$, such that

$$0 \in \partial^c_{\varepsilon^*} G(z) + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S, z), \quad g_\nu(z) > 0,\ \nu \in \Omega(\lambda),$$

where

$$\sqrt{\varepsilon^*} + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sum_{i=1}^{m} \xi_i \sqrt{\varepsilon_i}.$$

By (3.2.1), we get

$$\partial^c_{\varepsilon^*} G(x) \subseteq \sum_{i=1}^{m} \xi_i \partial^c_{\varepsilon^*} f_i(x).$$

The desired result follows. □

**Corollary 3.1.** *There exists an almost weakly quasi $\varepsilon$-Pareto solution for* (MP) *and $\varepsilon^*, \hat{\varepsilon}, \dot{\varepsilon}_\nu, \lambda_\nu \geq 0$, $\nu \in I(z)$, satisfying* (3.12) *and* (3.13)*.*

*Proof.* The conclusion follows by combining Lemma 3.3 and Theorem 3.10. □

**Example 3:**

Let us consider the following problem:

$$(\mathrm{P}_1) \quad \begin{array}{ll} \mathrm{Min} & (\sin x \cos y, \cos x \sin y) \\ \text{s.t.} & x + y - \frac{\sqrt{3}-1}{2} \geq 0, \\ & y \geq 0. \end{array}$$

Set $f_1(x, y) = \sin x \cos y,\ f_2(x, y) = \cos x \sin y,\ g(x, y) = \frac{\sqrt{3}-1}{2} - x - y$, and $S = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Take $(\xi_1, \xi_2) = (\frac{1}{2}, \frac{1}{2})$.

Using the weighted Manhattan metric method (also the weighted sum method) for $(\mathrm{P}_1)$, we get the scalar function $\frac{1}{2} \sin x \cos y + \frac{1}{2} \cos x \sin y$, i.e., $f(x, y) := \frac{1}{2} \sin(x + y)$. Hence, we consider the equivalent following problem:

$$(\mathrm{SP}) \quad \begin{array}{ll} \mathrm{Min} & \sin(x + y) \\ \text{s.t.} & \frac{\sqrt{3}-1}{2} - x - y \leq 0, \\ & y \geq 0. \end{array}$$

For a given vector $\varepsilon = (\varepsilon_1, \varepsilon_2) = (\frac{3}{4}, \frac{3}{4})$, we get $\sqrt{\theta} = \xi_1\sqrt{\varepsilon_1} + \xi_2\sqrt{\varepsilon_2} = \frac{\sqrt{3}}{2}$. Also, we obtain the relaxation of the feasible set of (P) with the tolerance $\sqrt{\theta}$ as

$$A_\theta = \{(x, y) \mid x + y + \frac{1}{2} \geq 0,\ y \geq 0\}.$$

It is clear that $(x, y) = (0, 0)$ does not belong to the feasible set $A$ of the problem above, but $(0, 0) \in A_\theta$. We will show that it is an almost quasi $\theta$-solution for (SP) satifying optimality condition introduced in Theorem 3.10, i.e., an almost quasi $\varepsilon$-Pareto solution for $(\mathrm{P}_1)$.

A simple computation gives:

$$\begin{aligned}
\partial^c f(0,0) = &\quad \{(u, v) \mid ud_1 + vd_2 \leq d_1 + d_2,\ \forall (d_1, d_2) \in \mathbb{R}^2\},\\
\partial^c g(0,0) = &\quad \{(-1, -1)\},\\
N(S, (0,0)) = &\quad \{(u, v) \mid u = 0,\ v \leq 0\}.
\end{aligned}$$

Let $\varepsilon^* = \dot{\varepsilon} = \frac{1}{16}$ and $\hat{\varepsilon} = 1 - \frac{\sqrt{3}}{2}$. We can check that

$$\begin{aligned}
(\tfrac{7}{8}, \tfrac{7}{8}) &= (1, 1) + \tfrac{1}{4}(-\tfrac{1}{2}, -\tfrac{1}{2}) \in \partial^c f(0,0) + \sqrt{\varepsilon^*}\mathcal{B} = \partial^c_{\varepsilon^*} f(0,0),\\
(-\tfrac{7}{8}, -\tfrac{7}{8}) &= (-1, -1) + \tfrac{1}{4}(\tfrac{1}{2}, \tfrac{1}{2}) \in \partial^c g(0,0) + \sqrt{\dot{\varepsilon}}\mathcal{B} = \partial^c_{\dot{\varepsilon}} g(0,0),\\
(0, 0) &= (0, \tfrac{1-\sqrt{3}}{2}) + (\tfrac{\sqrt{3}-1}{2})(0, 1) \in N(S, (0,0)) + \sqrt{\hat{\varepsilon}}\mathcal{B} = N_{\hat{\varepsilon}}(S, (0,0)).
\end{aligned}$$

If we choose $\lambda = 1$, then

$$(0, 0) \in \partial^c_{\varepsilon^*} f(0,0) + \lambda \partial^c_{\dot{\varepsilon}} g(0,0) + N_{\hat{\varepsilon}}(S, (0,0)).$$

Furthermore, since $\sqrt{\theta} = \sqrt{\varepsilon^*} + \lambda\sqrt{\dot{\varepsilon}} + \sqrt{\hat{\varepsilon}}$ and $0 < g(0,0) \leq \sqrt{\theta}$, the point $(0, 0)$ is an almost $\theta$-quasi solution for (SP). Hence, $(0, 0)$ is an almost weakly quasi $\varepsilon$-Pareto solution for $(\mathrm{P}_1)$.

3.2.2. *Weighted Tchebycheff Metric Method.* In formula (3.11), for $p = \infty$, we have a metric of Tchebycheff type. Suppose that $y_i = \min_{x \in F} f_i(x)$ for each $i \in M$, we consider the following weighted Tchebycheff problem:

$$\begin{array}{lll}
(\mathrm{WTP}_1) & \text{Minimize} & \max\limits_{1 \leq i \leq m} \xi_i(f_i(x) - y_i)\\
& \text{s.t.} & x \in F,
\end{array}$$

where the weighted vector $\xi \in \Sigma$, and $F, f_i, i \in M$, are as above. The method used to formulate the problem $(\mathrm{WTP}_1)$ is called a nonscalarizing method. For the details of the weighted Tchebycheff optimization technique, readers may refer to [5] (see also [35, Section 4.5]).

In the following results, we take

$$\theta' := \left(\min_{i \in M} \xi_i \sqrt{\varepsilon_i}\right)^2,$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_{++}$.

**Definition 3.8.** *A point $z \in F_{\theta'}$ is called an almost quasi $\theta'$-solution for $(\mathrm{WTP}_1)$ if there is no $x \in F$ such that*

$$\max_{1 \leq i \leq m} \xi_i(f_i(x) - y_i) + \sqrt{\theta'}\|x - z\| < \max_{1 \leq i \leq m} \xi_i(f_i(z) - y_i).$$

If the point $z \in F$, the word "almost" in the definition above can be removed.

**Proposition 3.2.** *If $z \in F_{\theta'}$ is an almost quasi $\theta'$-solution for $(\mathrm{WTP}_1)$, then $z$ is an almost weakly quasi $\varepsilon$-Pareto solution for* (MP).

*Proof.* Suppose that $z$ is an almost quasi $\theta'$-solution for $(\mathrm{WTP}_1)$ and that $z$ is not an almost weakly quasi $\varepsilon$-Pareto solution for (MP). Then, there exists $x' \in F$ such that

$$f_i(x') + \sqrt{\varepsilon_i}\|x' - z\| < f_i(z), \quad \forall i \in M.$$

For any $\xi_i \in (0,1)$, we get

$$\xi_i f_i(x') + \xi_i\sqrt{\varepsilon_i}\|x' - z\| < \xi_i f_i(z), \quad \forall i \in M.$$

Hence,

$$\xi_i[f_i(x') - y_i] + \xi_i\sqrt{\varepsilon_i}\|x' - z\| < \xi_i[f_i(z) - y_i], \quad \forall i \in M.$$

Thus,

$$\max_{i\in M}\{\xi_i[f_i(x') - y_i] + \xi_i\sqrt{\varepsilon_i}\|x' - z\|\} < \max_{i\in M}\xi_i[f_i(z) - y_i].$$

Since $\sqrt{\theta'} \leq \xi_i\sqrt{\varepsilon_i}$ for all $i \in M$, we get

$$\max_{i\in M}\left\{\xi_i[f_i(x') - y_i] + \sqrt{\theta'}\|x' - z\|\right\} < \max_{i\in M}\xi_i[f_i(z) - y_i],$$

a contradiction. The proof is complete. □

**Example 4:** This example illustrates Proposition 3.2.

i) *Using a nonscalarizing method:* Let us consider the following problem

$$(\mathrm{P}_2) \quad \begin{array}{ll} \mathrm{Min} & (|x+y|, |x-y|) \\ \text{s.t.} & \frac{1}{4} - x - y \leq 0, \\ & y \geq 0. \end{array}$$

For $F := \{(x,y) \mid 1/4 - x - y \leq 0, y \geq 0\}$, we can check that $y_1 = \min_F |x+y| = 1/4$ at all points satisfying $x + y = 1/4, y \geq 0$, and $y_2 = \min_F |x-y| = 0$ at all points $(x,y)$ such that $x = y = 1/8$.

Given $\varepsilon = (\frac{1}{4}, \frac{3}{4}), \xi = (\frac{1}{2}, \frac{1}{2})$, take $\sqrt{\theta'} = \min\left\{\xi_1\sqrt{\varepsilon_1}, \xi_2\sqrt{\varepsilon_2}\right\} = \min\left\{\frac{1}{2}\sqrt{\frac{1}{4}}, \frac{1}{2}\sqrt{\frac{3}{4}}\right\} = \frac{1}{4}$, we get

$$F_{\theta'} = \{(x,y) \mid x + y \geq 0, y \geq 0\}.$$

Based on the model $(\mathrm{WTP}_1)$, by using the ideal point $(1/4, 0)$, we have the following problem

$$(\mathrm{P}_2') \quad \begin{array}{lll} \text{Minimize} & \max\limits_{1\leq i\leq m} & \left[\frac{1}{2}(|x+y| - 1/4), \frac{1}{2}|x-y|\right] \\ \text{s.t.} & & (x,y) \in F. \end{array}$$

It is easy to see that $(0,0)$ is not in the feasible set $F$, but $(0,0) \in F_{\theta'}$. Firstly, we check that $z = (0,0)$ is an almost quasi $\theta'$- solution of $(\mathrm{P}_2')$, i.e, there is no $x \in F$ such that

$$\max_{i=1,2}\xi_i(f_i(x) - y_i) + \sqrt{\theta'}\|x - z\| < \max_{i=1,2}\xi_i(f_i(z) - y_i). \tag{3.14}$$

Note that the right hand side of (3.14) is

$$\max\left(\frac{1}{2}(-1/4), \frac{1}{2}.0\right) = 0,$$

and the left hand side is

$$\max\left(\frac{1}{2}(|x+y| - 1/4) + \frac{1}{4}\|x - z\|, \frac{1}{2}|x-y| + \frac{1}{4}\|x - z\|\right) \geq 0, \forall (x,y) \in F.$$

Hence, there is no $(x,y) \in F$ such that (3.14) is satisfied, i.e., $z = (0,0)$ is an almost $\theta'$- solution for $(\mathrm{P}_2')$. We now check that $z = (0,0)$ is an almost weakly quasi $\varepsilon$-Pareto

solution for $(\mathrm{P}_2)$ with $\varepsilon = (\frac{1}{4}, \frac{3}{4})$. Indeed, it is obvious that there exists no $(x, y) \in F$ such that

$$\begin{cases} |x+y| + \frac{1}{2}\|(x,y)\| < 0, \\ |x-y| + \frac{\sqrt{3}}{2}\|(x,y)\| < 0. \end{cases}$$

Hence, according to Definition 3.5, the point $z = (0, 0)$ is an almost weakly quasi $\varepsilon$-Pareto solution for $(\mathrm{P}_2)$.

ii) *Using a scalarizing method:* In this part we deal with almost weakly quasi $\varepsilon$-Pareto solution for $(\mathrm{P}_2)$ using a scalarizing method. From the bi-objective problem $(\mathrm{P}_2)$ and the weighted vector $\xi = (\frac{1}{2}, \frac{1}{2})$, we get a scalar problem with the objective function is $\frac{1}{2}|x+y| + \frac{1}{2}|x-y|)$ based on the weighted compromising method with Manhattan metric. Hence, we consider the following problem:

$$\begin{array}{lll} (\mathrm{P}_2'') & \text{Min} & |x+y| + |x-y|) \\ & \text{s.t.} & \frac{1}{4} - x - y \leq 0, \\ & & y \geq 0. \end{array}$$

Obviously, $z = (0, 0)$ does not belong to the feasible set of $(\mathrm{P}_2'')$. Set $F(x, y) = |x + y| + |x - y|$. For all the feasible solutions $(x, y)$ of $(\mathrm{P}_2'')$, it is clear that

$$F(0,0) \leq F(x,y) + \sqrt{\theta'}\|(x,y) - (0,0)\|.$$

Hence, $(0, 0)$ is an almost quasi $\theta'$-solution of $(\mathrm{P}_2'')$ by Definition 2.4 (ii). Therefore, it is an almost weakly quasi $\varepsilon$-Pareto solution for $(\mathrm{P}_2)$ according to Definition 3.5.

Inspired by a result given by Choo and Atkins [16, Theorem 3.1], we are concerned with the notion of utopia points for (MP) [35, Definition 4.23]. For the ideal point of (MP), the utopia point is denoted and defined by

$$y^U := y^I - \varepsilon, \quad \text{where } \varepsilon \in \mathbb{R}^m_{++} \text{ has small components.} \tag{3.15}$$

It is clear that, for any $x \in F$, we have $f(x) > y^U$.

We also note that, for a weighted vector $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_{++}$, if we set $p = \sum_{i=1}^m \xi_i$, then we get $\frac{\xi}{p} \in \Sigma$. Hence, we can deal with a weighted vector $\xi \in \mathbb{R}^m_{++}$ instead of $\xi \in \Sigma$.

The following proposition is inspired by [35, Theorem 4.24]. Now, instead of the problem $(\mathrm{WTP}_1)$, we consider the problem $(\mathrm{WTP}_2)$, defined by

$$\begin{array}{lll} (\mathrm{WTP}_2) & \text{Minimize} & \max\limits_{1 \leq i \leq m} \xi_i(f_i(x) - y_i^U) \\ & \text{s.t.} & x \in F. \end{array}$$

**Proposition 3.3.** *Suppose that $z \in F_{\theta'}$ is an almost weakly quasi $\varepsilon$-Pareto solution for* (MP)*. If the weighted vector $\xi$ is chosen by*

$$\xi_i := \frac{1}{f_i(z) - y_i^U}, \quad i = 1, \ldots, m, \tag{3.16}$$

*then $z$ is an almost quasi $\theta'$-solution for* $(\mathrm{WTP}_2)$.

*Proof.* Let $z$ be an almost weakly quasi $\varepsilon$-Pareto solution for (MP). Let us choose $\xi \in \mathbb{R}^m_{++}$ such that (3.16) is satisfied.

Suppose, to the contrary, that $z$ is not an almost quasi $\theta'$-solution for $(\mathrm{WTP}_2)$. Then there exists $x' \in F$ such that

$$\max_{1 \leq i \leq m} \xi_i(f_i(x') - y_i^U) + \sqrt{\theta'}\|x' - z\| < \max_{1 \leq i \leq m} \xi_i(f_i(z) - y_i^U).$$

Hence,

$$\xi_i(f_i(x') - y_i^U) + \xi_i\sqrt{\varepsilon_i}\|x' - z\| < 1, \quad \forall i \in M.$$

This implies that

$$(f_i(x') - y_i^U) + \sqrt{\varepsilon_i}\|x' - z\| < \frac{1}{\xi_i} = f_i(z) - y_i^U.$$

Hence,

$$f_i(x') + \sqrt{\varepsilon_i}\|x' - z\| < f_i(z), \quad \forall i \in M,$$

which contradicts the fact that $z$ is an almost weakly quasi $\varepsilon$-Pareto solution for (MP). The desired result follows. □

**Remark 3.3.** In Propositions 3.2 and 3.3, if we take $z \in F$, then the word "almost" in the propositions is removed.

We now study approximate solutions for $(\mathrm{WTP}_2)$. Suppose that the weighted vector $\xi$ is chosen by (3.16), where $y_i^U$ is determined by (3.15). Set

$$h_i(x) := \xi_i(f_i(x) - y_i^U), \quad i = 1, \ldots, m. \tag{3.17}$$

We get $h_i(x) > 0$ if $x \in F$ for all $i \in M$. Set

$$H(x) := \max_{i \in M} h_i(x) \tag{3.18}$$

and $M_H(z) := \{i \in M \mid H(z) = h_i(z)\}$. Then the problem $(\mathrm{WTP}_2)$ can be written in the form:

$$\begin{array}{lll} (\mathrm{WTP}_2') & \text{Minimize} & H(x) \\ & \text{s.t.} & g_\nu(x) \le 0,\ \nu \in \Omega, \\ & & x \in S. \end{array} \tag{3.19}$$

**Lemma 3.4.** *Suppose that $z \in F$ is a quasi $\theta'$-solution for $(\mathrm{WTP}_2')$, and that the condition $(\mathcal{H})$ holds for $z$. Then, there exist $\varepsilon^*, \hat{\varepsilon}, \dot{\varepsilon}_\nu, \lambda_\nu \ge 0$, $\nu \in \Omega(\lambda)$, such that*

$$0 \in \partial_{\varepsilon^*}^c H(z) + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \partial_{\dot{\varepsilon}_\nu/\lambda_\nu^2}^c g_\nu(z) + N_{\hat{\varepsilon}}^Q(S, z),$$

*with*

$$g_\nu(z) = 0,\ \nu \in \Omega(\lambda), \quad \sqrt{\varepsilon^*} + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta'}.$$

*Proof.* Since $z$ is a quasi $\theta'$-solution of the problem $(\mathrm{WTP}_2')$, and the condition $(\mathcal{H})$ is satisfied for $z$, the desired result follows from [28, Corollary 3.2 iv)]. □

Similar to Theorem 3.10, we can state the following theorem regarding the existence of almost quasi $\varepsilon$-solutions for $(\mathrm{WTP}_2')$. Its proof is omitted.

**Theorem 3.11.** *There exists $z \in F_{\theta'}$ that is an almost regular $\theta'$-solution for $(\mathrm{WTP}_2')$, and there exist $\varepsilon^*, \hat{\varepsilon}, \dot{\varepsilon}_\nu, \lambda_\nu \ge 0$, $\nu \in \Omega(\lambda)$, such that*

$$0 \in \partial_{\varepsilon^*}^c H(z) + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \partial_{\dot{\varepsilon}_\nu}^c g_\nu(z) + N_{\hat{\varepsilon}}^Q(S, z), \quad g_\nu(z) > 0,\ \forall \nu \in \Omega(\lambda). \tag{3.20}$$

*where*

$$\sqrt{\varepsilon^*} + \sum_{\nu \in \Omega(\lambda)} \lambda_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta'}. \tag{3.21}$$

We also note that, by Theorem 2.2 and [33, Propistion 2.3.12], we get

$$\begin{aligned}\partial^c_{\varepsilon^*} H(z) &\subseteq \operatorname{co}\left\{\bigcup_{i\in M(z)} \partial^c h_i(z)\right\} + \sqrt{\varepsilon^*}B \\ &\subseteq \sum_{i\in M(z)} \zeta_i \partial^c_{\varepsilon^*} h_i(z),\end{aligned}$$

where $\sum_{i\in M(z)} \zeta_i = 1$, $\zeta_i > 0$. Hence, (3.20) and (3.21) can be rewritten by

$$0 \in \sum_{i\in M(z)} \zeta_i \partial^c_{\varepsilon^*} h_i(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\dot\varepsilon_\nu} g_\nu(z) + N^Q_{\hat\varepsilon}(S,z), \quad g_\nu(z) > 0,\ \forall \nu \in \Omega(\lambda). \tag{3.22}$$

and

$$\sqrt{\varepsilon^*} + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \sqrt{\dot\varepsilon_\nu} + \sqrt{\hat\varepsilon} = \sqrt{\theta'}. \tag{3.23}$$

From Proposition 3.2 and Theorem 3.11, we propose the following theorem.

**Theorem 3.12.** *There exists an almost weakly quasi $\varepsilon$-Pareto solution $z$ for* (MP)*, and there exist $\bar\varepsilon_i \geq 0$, $i = 1,\ldots,m$, and $\mu_i \geq 0$, $i = 1,\ldots,m$; $\hat\varepsilon \geq 0$, $\dot\varepsilon_\nu \geq 0$, and $\lambda_\nu \geq 0$ for all $\nu \in \Omega$ such that*

$$0 \in \sum_{i\in M} \mu_i \partial^c_{\bar\varepsilon_i} f_i(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\dot\varepsilon_\nu} g_\nu(z) + N^Q_{\hat\varepsilon}(S,z), \quad g_\nu(z) > 0, \forall \nu \in \Omega(\lambda). \tag{3.24}$$

*and*

$$\sum_{i\in M} \mu_i \sqrt{\bar\varepsilon_i} + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \sqrt{\dot\varepsilon_\nu} + \sqrt{\hat\varepsilon} = \sqrt{\theta'}. \tag{3.25}$$

*Proof.* Based on Proposition 3.2 and Theorem 3.11, there exists an almost weakly quasi $\varepsilon$-Pareto solution for (MP) satisfying the conditions (3.20) and (3.21).

By Theorem 2.2 and [33, Propostion 2.3.12], we get

$$\begin{aligned}\partial^c_{\varepsilon^*} H(z) &\subseteq \operatorname{co}\left\{\bigcup_{i\in M(z)} \partial^c h_i(z)\right\} + \sqrt{\varepsilon^*}B \\ &\subseteq \sum_{i\in M(z)} \zeta_i \partial^c_{\varepsilon^*} h_i(z),\end{aligned}$$

where $\sum_{i\in M(z)} \zeta_i = 1$, $\zeta_i > 0$.

Thus, we obtain

$$0 \in \sum_{i\in M(z)} \zeta_i \partial^c_{\varepsilon^*} h_i(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\dot\varepsilon_\nu} g_\nu(z) + N^Q_{\hat\varepsilon}(S,z), \tag{3.26}$$

with $g_\nu(z) > 0$ for all $\nu \in \Omega(\lambda)$, and

$$\sqrt{\varepsilon^*} + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \sqrt{\dot\varepsilon_\nu} + \sqrt{\hat\varepsilon} = \sqrt{\theta'}. \tag{3.27}$$

Furthermore,

$$\begin{aligned}\partial^c_{\varepsilon^*} h_i(z) &= \partial^c_{\varepsilon^*}(\xi_i(f_i(z) - y_i)) \\ &= \xi_i \partial^c_{\frac{\varepsilon^*}{\xi_i^2}}(f_i(z) - y_i) \\ &\subseteq \xi_i \partial^c_{\frac{\varepsilon^*}{\xi_i^2}} f_i(z).\end{aligned}$$

Set $\bar\varepsilon_i = \frac{\varepsilon^*}{\xi_i^2}$ for each $i \in M$. Then $\partial^c_{\varepsilon^*} h_i(z) \subseteq \xi_i \partial^c_{\bar\varepsilon_i} f_i(z)$.

Substituting into (3.26), we obtain

$$0 \in \sum_{i\in M(z)} \xi_i\zeta_i\partial^c_{\bar{\varepsilon}_i} f_i(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S,z), \quad g_\nu(z) > 0, \forall \nu \in \Omega(\lambda).$$

Finally, set $\mu_i = \xi_i\zeta_i$ for $i \in M(z)$ and $\mu_i = 0$ for $i \in M \setminus M(z)$, and obtain

$$0 \in \sum_{i\in M} \mu_i\partial^c_{\bar{\varepsilon}_i} f_i(z) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\dot{\varepsilon}_\nu} g_\nu(z) + N^Q_{\hat{\varepsilon}}(S,z), \quad g_\nu(z) > 0, \forall \nu \in \Omega(\lambda).$$

In addition, since $\sqrt{\varepsilon^*} = \xi_i\sqrt{\bar{\varepsilon}_i}$ for each $i \in M$, from (4.35) we get

$$\sqrt{\varepsilon^*} = \sum_{i=1}^m \zeta_i\xi_i\sqrt{\bar{\varepsilon}_i} = \sum_{i=1}^m \mu_i\sqrt{\bar{\varepsilon}_i}.$$

Hence,

$$\sum_{i\in M} \mu_i\sqrt{\bar{\varepsilon}_i} + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu\sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta'}.$$

□

**Example 5:**

Let us reconsider the problem $(\mathrm{P}_2)$:

$$(\mathrm{P}_2) \quad \begin{array}{ll} \mathrm{Min} & (|x+y|, |x-y|) \\ \text{s.t.} & -x-y+\frac{1}{4} \le 0, \\ & y \ge 0. \end{array}$$

By using an utopia point and associated the problem $(\mathrm{WTP}_2)$, we will show that $z = (0,0)$ is a weakly quasi $\varepsilon$-Pareto solution to the problem $(\mathrm{P}_2)$ satisfying Theorem 3.12.

Note that $g(0,0) > 0$, i.e., the $(0,0)$ point does not belong to the feasible set of $(\mathrm{P}_2)$. Moreover, it is an almost weakly quasi $\varepsilon$-Pareto solution for $(\mathrm{P}_2)$ with $\varepsilon = (\frac{1}{4}, \frac{3}{4})$. We will check that $z$ satisfies optimality condition given by (3.24) and (3.25).

Let $f_1(x,y) = |x+y|, f_2(x,y) = |x-y|, g(x,y) = -x-y+1/4, S := \{(x,y) \mid y \ge 0\}$ and $\bar{z} = (0,0)$. A simple computation gives that

$\partial^c_{\bar{\varepsilon}_1} f_1(\bar{z}) = \partial^c f_1(0,0) + \sqrt{\bar{\varepsilon_1}}B = \{(t,t) \mid -1 \le t \le 1\} + \frac{1}{2}B,$

$\partial^c_{\bar{\varepsilon}_2} f_2(\bar{z}) = \partial^c f_2(0,0) + \sqrt{\bar{\varepsilon_2}}B = \{(t,-t) \mid -1 \le t \le 1\} + \frac{\sqrt{3}}{2}B.$

$\partial^c_{\hat{\varepsilon}} g(\bar{z}) = \partial^c g(0,0) + \sqrt{\hat{\varepsilon}}B = \{(-1,-1) + \sqrt{\hat{\varepsilon}}B.$

$N^Q_{\bar{\varepsilon}}(S,\bar{z}) = N(S,\bar{z}) + \sqrt{\bar{\varepsilon}}B = \{(u,v) \mid u = 0, v \le 0\} + \sqrt{\bar{\varepsilon}}B.$

Choose the utopia point $(y_1^U, y_2^U) = (-1,-1)$. We get $\xi_1 = \frac{1}{f_1(z)-y_1^U} = 1$, $\xi_2 = \frac{1}{f_2(z)-y_2^U} = 1$ Hence,

$$\sqrt{\theta'} = \min_{i=1,2} \xi_i\sqrt{\varepsilon_i} = \min(1.\frac{1}{2}, 1.\frac{\sqrt{3}}{2}) = \frac{1}{2}.$$

It is obvious that $\bar{z} = (0,0) \in F_{\theta'}$. The conditions (3.24) and (3.25) in this case are as follows:

$$0 \in \mu_1\partial^c_{\bar{\varepsilon}_1} f_1(0,0) + \mu_2\partial^c_{\bar{\varepsilon}_2} f_2(0,0) + \lambda\partial^c_{\hat{\varepsilon}} g(0,0) + N^Q_{\hat{\varepsilon}}(S,(0,0)). \tag{3.28}$$

and

$$\mu_1.\frac{1}{2} + \mu_2\frac{\sqrt{3}}{2} + \lambda\sqrt{\dot{\varepsilon}} + \sqrt{\hat{\varepsilon}} = \frac{1}{2}. \tag{3.29}$$

It is clear that (3.29) holds with $\mu_1 = \frac{1}{4}, \mu_2 = \frac{1}{4\sqrt{3}}, \lambda = \frac{1}{2}, \sqrt{\hat{\varepsilon}} = \frac{1}{4}$ and $\sqrt{\hat{\varepsilon}} = \frac{1}{8}$. In addition, the inclusion (3.28) becomes

$$0 \in \frac{1}{4}\partial^c_{1/4} f_1(0,0) + \frac{1}{4\sqrt{3}}\partial^c_{3/4} f_2(0,0) + \frac{1}{2}\partial^c_{1/16} g(0,0) + N^Q_{1/64}(S,(0,0)).$$

It is equivalent to

$$0 \in \frac{1}{4}\partial^c f_1(0,0) + \frac{1}{4\sqrt{3}}\partial^c f_2(0,0) + \frac{1}{2}\partial^c g(0,0) + N(S,(0,0)) + \frac{1}{2}B. \tag{3.30}$$

This inclusion holds with $(1,1) \in \partial^c f_1(0,0), (0,0) \in \partial^c f_2(0,0), (-1,-1) \in \partial^c g(0,0), (0,0) \in N(S,(0,0))$ and $(1/2, 1/2) \in B$. Hence, (3.28) and (3.29) are satisfied.

## 4. Approximate Duality Theorems

In this section, the relationships between the multi-objective problem (MP) and its dual problem are investigated via a corresponding pair of single-objective problems.

For $\lambda \in \mathbb{R}^{(\Omega)}_+$, let us consider the Lagrange vector function associated with (MP), defined by

$$\mathcal{L}(y,\lambda) := \big(f_1(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y), \ldots, f_m(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y)\big).$$

From the formulation (3.22), we propose a dual problem for (MP) in Wolfe type as follows:

$$\begin{array}{lll} \text{(MD) Max} & \mathcal{L}(y,\lambda) & \\ \text{s.t.} & 0 \in \displaystyle\sum_{i\in M} \mu_i \partial^c_{\alpha_i} f_i(y) + \sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\beta_\nu} g_\nu(y) + N^Q_\gamma(S,y), & \\ & \mu \in \mathbb{R}^m_+,\ \alpha \in \mathbb{R}^m_+,\ \beta_\nu \in \mathbb{R}_+,\ \nu \in \Omega(\lambda),\ \gamma \in \mathbb{R}_+, & \\ & (y,\lambda) \in S \times \mathbb{R}^{(\Omega)}_+. & \end{array} \tag{4.31}$$

We denote by $F_{MD}$ the feasible set of (MD).

**Definition 4.9.** *Given $\varepsilon \in \mathbb{R}^m_{++}$ and $\sqrt{\theta} = \min_{i\in M} \xi_i \sqrt{\varepsilon_i}$, a point $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in F_{MD}$ is said to be a weakly quasi $\varepsilon$-Pareto solution for* (MD) *if there does not exist $(y, \lambda, \mu, \alpha, \beta, \gamma) \in F_{MD}$ such that*

$$f_i(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) > f_i(\bar{y}) + \sum_{\nu\in\Omega} \bar{\lambda}_\nu g_\nu(\bar{y}) + \sqrt{\theta}\|y - \bar{y}\| + \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1, \quad \forall i \in M. \tag{4.32}$$

From the problem $(\text{WTP}'_2)$ given by (3.19), we formulate its dual problem in Wolfe type as follows:

$$\begin{array}{lll} \text{(WTD)} \quad \text{Max} & H(y) + \displaystyle\sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) & \\ \text{s.t.} & 0 \in \partial^c_\alpha H(y) + \displaystyle\sum_{\nu\in\Omega(\lambda)} \lambda_\nu \partial^c_{\beta_\nu} g_\nu(y) + N^Q_\gamma(S,y), & \\ & (y,\lambda) \in S \times \mathbb{R}^{(\Omega)}_+. & \end{array}$$

We denote by $F_D$ the feasible set of (WTD).

**Lemma 4.5.** *If $(y, \lambda, \alpha, \beta, \gamma) \in F_D$ is a feasible solution of* (WTD)*, then there exists $\mu \in \mathbb{R}^m_+$ such that $(y, \lambda, \mu, \alpha, \beta, \gamma)$ is a feasible solution of* (MD).

*Proof.* Note that $\partial^c_\alpha H(y) = \partial^c H(y) + \sqrt{\alpha}B$. Since $H(y) := \max_{i\in M} h_i(y)$, we have $\partial^c H(y) \subseteq \text{co}\{\partial^c h_i(y) \mid i \in M_H(y)\}$. There exists $\mu \in \mathbb{R}^m_+$ with $\sum_{i\in M_H(y)} \mu_i = 1$ and $\mu_i = 0$ if $i \in M \setminus M_H(y)$ such that

$$\partial^c_\alpha H(y) \subseteq \sum_{i\in M} \mu_i \partial^c h_i(y) + \sqrt{\alpha}B = \sum_{i\in M} \mu_i[\partial^c h_i(y) + \sqrt{\alpha}B] = \sum_{i\in M} \mu_i \partial^c_\alpha h_i(y).$$

By comparing the constructions of problems (WTD) and (MD), we obtain the conclusion of the lemma. □

**Definition 4.10.** *Given $\varepsilon \in \mathbb{R}^m_{++}$ and $\sqrt{\theta} = \min_{i\in M} \xi_i\sqrt{\varepsilon_i}$, a point $(\bar{y}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in F_D$ is said to be a quasi $\theta$-solution for* (WTD) *if for any $(y, \lambda, \alpha, \beta, \gamma) \in F_D$, the following inequality holds:*

$$H(\bar{y}) + \sum_{\nu\in\Omega} \bar{\lambda}_\nu g_\nu(\bar{y}) \geq H(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) - \sqrt{\theta}\|y - \bar{y}\| - \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1. \tag{4.33}$$

We are now in a position to propose a scheme for investigating the relationship between (MP) and (MD), using the diagram given in the first section.

**Lemma 4.6.** *If $(\bar{y}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in F_D$ is a quasi $\theta$-solution for* (WTD)*, then there exists $\bar{\mu} \in \Sigma$ such that $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is a weakly quasi $\varepsilon$-Pareto solution for* (MD)*.*

*Proof.* Let $(\bar{y}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in F_D$ be a quasi $\varepsilon$-solution for (WTD). By Lemma 4.5, there exists $\bar{\mu} \in \Sigma$, $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ such that it is a feasible solution of (MD). Suppose, to the contrary, that $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is not a weakly quasi $\varepsilon$-Pareto solution for (MD). Then there exists $(y, \lambda, \mu, \alpha, \beta, \gamma) \in F_{MD}$ such that

$$f_i(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) > f_i(\bar{y}) + \sum_{\nu\in\Omega} \bar{\lambda}_\nu g_\nu(\bar{y}) + \sqrt{\theta}\|y - \bar{y}\| + \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1, \quad \forall i \in M. \tag{4.34}$$

This implies

$$H(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) > H(\bar{y}) + \sum_{\nu\in\Omega} \bar{\lambda}_\nu g_\nu(\bar{y}) + \sqrt{\theta}\|y - \bar{y}\| + \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1,$$

a contradiction to (4.33). □

**Lemma 4.7.** *Suppose that $H(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y)$ is $\theta$-pseudoconvex. If $z \in F_\theta$ is an almost quasi $\theta$-solution for $(\text{WTP}'_2)$ satisfying the conditions* (3.20)–(3.21)*, then there exist $\bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}$ such that $(z, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in F_D$ and it is a quasi $\theta$-solution for* (WTD)*.*

*Proof.* Let $z \in F_\theta$ be an almost quasi $\theta$-solution for $(\text{WTP}'_2)$ satisfying the conditions (3.20)–(3.21). From these conditions, set $\bar{\lambda} := \lambda$, $\bar{\alpha} := \varepsilon^*$, $\bar{\beta} := \dot{\varepsilon}$, and $\bar{\gamma} := \hat{\varepsilon}$. Hence,

$$0 \in \partial^c_{\bar{\alpha}} H(z) + \sum_{\nu\in\Omega(\bar{\lambda})} \bar{\lambda}_\nu \partial^c_{\bar{\beta}_\nu} g_\nu(z) + N^Q_{\bar{\gamma}}(S, z), \quad g_\nu(z) > 0, \forall \nu \in \Omega(\bar{\zeta}). \tag{4.35}$$

and

$$\sqrt{\bar{\alpha}} + \sum_{\nu\in\Omega(\lambda)} \bar{\lambda}_\nu \sqrt{\bar{\beta}_\nu} + \sqrt{\bar{\gamma}} = \sqrt{\theta}.$$

We get $(z, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in F_D$. It remains to prove that

$$H(z) + \sum_{\nu\in\Omega} \bar{\lambda}_\nu g_\nu(z) \geq H(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) - \sqrt{\theta}\|y - \bar{y}\| - \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1.$$

Set $\mathcal{H}(y, \lambda) := H(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y)$. We claim that

$$\mathcal{H}(z, \bar{\lambda}) \geq \mathcal{H}(z, \lambda) - \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1, \tag{4.36}$$

i.e.,

$$\sum_{\nu\in\Omega}(\bar{\lambda}_\nu-\lambda_\nu)g_\nu(z)\geq-\sqrt{\theta}\|\lambda-\bar{\lambda}\|_1.$$

Indeed, write $\Omega=\Omega(\bar{\lambda})\cup(\Omega\setminus\Omega(\bar{\lambda}))$. If $\nu\in\Omega(\bar{\lambda})$, then $g_\nu(z)>0$. Hence,

$$(\bar{\lambda}_\nu-\lambda_\nu)g_\nu(z)\geq-|\bar{\lambda}_\nu-\lambda_\nu|g_\nu(z).$$

Since $z\in F_\theta$, $-g_\nu(z)\geq-\sqrt{\theta}$ for all $\nu\in\Omega(\bar{\lambda})$. Therefore,

$$(\bar{\lambda}_\nu-\lambda_\nu)g_\nu(z)\geq-\sqrt{\theta}|\bar{\lambda}_\nu-\lambda_\nu|.$$

If $\nu\notin\Omega(\bar{\lambda})$, then $\bar{\lambda}_\nu=0$ and $g_\nu(z)\leq0$, so

$$(\bar{\lambda}_\nu-\lambda_\nu)g_\nu(z)\geq0.$$

Thus, inequality (4.36) is proven.

Next, we prove that $\mathcal{H}(z,\lambda)\geq\mathcal{H}(y,\lambda)-\sqrt{\theta}\|z-y\|$ for all $(y,\lambda,\alpha,\beta,\gamma)\in F_D$. Indeed, since

$$0\in\partial^c_\alpha H(y)+\sum_{\nu\in\Omega(\lambda)}\lambda_\nu\partial^c_{\beta_\nu}g_\nu(y)+N^Q_\gamma(S,y),$$

there exist $u\in\partial^c_\alpha H(y)$, $v_\nu\in\partial^c_{\beta_\nu}g_\nu(y)$ for $\nu\in\Omega(\lambda)$, and $w\in N^Q_\gamma(S,y)$ such that

$$u+\sum_{\nu\in\Omega(\lambda)}\lambda_\nu v_\nu=-w.$$

Hence,

$$H^c(y;x-y)+\sqrt{\alpha}\|x-y\|+\sum_{\nu\in\Omega(\zeta)}\lambda_\nu[g^c_\nu(y;x-y)+\sqrt{\beta_\nu}\|x-y\|]\geq-\sqrt{\gamma}\|x-y\|,\quad\forall x\in S,$$

i.e.,

$$H^c(y;x-y)+\sum_{\nu\in\Omega(\lambda)}\lambda_\nu g^c_\nu(y;x-y)+\sqrt{\theta}\|x-y\|\geq0,\quad\forall x\in S.$$

Since $\mathcal{H}$ is $\theta$-pseudoconvex, it follows that

$$\mathcal{H}(x,\lambda)+\sqrt{\theta}\|x-y\|\geq\mathcal{H}(y,\zeta),\quad\forall x\in S.$$

Thus,

$$\mathcal{H}(z,\lambda)\geq\mathcal{H}(y,\lambda)-\sqrt{\theta}\|z-y\|.$$

Combining this with (4.36), we obtain

$$\mathcal{H}(z,\bar{\lambda})\geq\mathcal{H}(y,\lambda)-\sqrt{\theta}\|z-y\|-\sqrt{\theta}\|\lambda-\bar{\lambda}\|_1.$$

We conclude that $(z,\bar{\lambda},\bar{\alpha},\bar{\beta},\bar{\gamma})\in F_D$ is a quasi $\theta$-solution for (WTD). □

Finally, we establish the relationship between (MP) and (MD).

**Theorem 4.13.** *Suppose that $z\in F_\theta$ is an almost weakly quasi $\varepsilon$-solution for* (MP) *and the conditions* (3.20) *and* (3.21) *holds for $z$. Suppose further that the assumption of Lemma 4.7 is satisfied. Then, there exists $\bar{\mu},\bar{\lambda},\bar{\alpha},\bar{\beta},\bar{\gamma}$ such that $(z,\bar{\mu},\bar{\lambda},\bar{\alpha},\bar{\beta},\bar{\gamma})$ is a weakly quasi $\varepsilon$-Pareto solution for* (MD).

*Proof.* Suppose that $z\in F_\theta$ is an almost weakly quasi $\varepsilon$-Pareto solution for (MP). If we choose $\xi$ as in the formulation given by (3.16), then, by Proposition 3.3, it is an almost quasi $\theta$-solution for the problem $(\mathrm{WTP}_2)$, also for the problem $(\mathrm{WTP}'_2)$.

Since the conditions (3.20) and (3.21) holds for $z$, we get

$$0\in\partial^c_{\varepsilon^*}H(z)+\sum_{\nu\in\Omega(\bar{\lambda})}\bar{\lambda}_\nu\partial^c_{\hat{\varepsilon}_\nu}g_\nu(z)+N^Q_{\hat{\varepsilon}}(S,z),\quad g_\nu(z)>0,\ \forall\nu\in\Omega(\bar{\lambda}).\tag{4.37}$$

where

$$\sqrt{\varepsilon^*} + \sum_{\nu\in\Omega(\bar{\lambda})} \bar{\lambda}_\nu \sqrt{\dot{\varepsilon}_\nu} + \sqrt{\hat{\varepsilon}} = \sqrt{\theta}.$$

Set $\bar{\alpha} = \varepsilon^*$ and $\bar{\beta} = (\bar{\beta}_\nu)$ such that $\bar{\beta}_\nu = \dot{\varepsilon}_\nu$ and $\bar{\gamma} = \hat{\varepsilon}$. Then, by Lemma 4.7, the point $(z, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is a quasi $\theta$-solution for (WTD). Note that

$$\begin{aligned}\partial^c_{\varepsilon^*} H(z) = \partial^c_{\bar{\alpha}} H(z) &\subseteq \operatorname{co}\left\{\bigcup_{i\in M(z)} \partial^c h_i(z)\right\} + \sqrt{\bar{\alpha}}B\\ &\subseteq \sum_{i\in M(z)} \zeta_i \partial^c_{\bar{\alpha}} h_i(z),\end{aligned}$$

where $\sum_{i\in M(z)} \zeta_i = 1$, $\zeta_i > 0$. Furthermore,

$$\begin{aligned}\partial^c_{\bar{\alpha}} h_i(z) &= \partial^c_{\bar{\alpha}}(\xi_i(f_i(z) - y_i))\\ &= \xi_i \partial^c_{\frac{\bar{\alpha}}{\xi_i^2}}(f_i(z) - y_i)\\ &\subseteq \xi_i \partial^c_{\frac{\bar{\alpha}}{\xi_i^2}} f_i(z).\end{aligned}$$

Set $\bar{\alpha}_i = \frac{\bar{\alpha}}{\xi_i^2}$ for each $i \in M$. Then $\partial^c_{\bar{\alpha}} h_i(z) \subseteq \xi_i \partial^c_{\bar{\alpha}_i} f_i(z)$. Hence,

$$\zeta_i \partial^c_{\bar{\alpha}} h_i(z) = \zeta_i \xi_i \partial^c_{\bar{\alpha}_i} f_i(z).$$

Set $\bar{\mu}_i = \zeta_i \xi_i, i \in M$. Combining this and (4.37), we obtain

$$0 \in \sum_{i\in M} \bar{\mu}_i \partial^c_{\bar{\alpha}_i} f_i(z) + \sum_{\nu\in\Omega(\bar{\lambda})} \bar{\lambda}_\nu \partial^c_{\bar{\beta}_\nu} g_\nu(z) + N^Q_{\bar{\gamma}}(S, z). \tag{4.38}$$

This means that $(z, \bar{\mu}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is a feasible solution of (MD).

We now prove that $(z, \bar{\mu}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is a weakly quasi $\varepsilon$-Pareto solution for (MD). Indeed, suppose to the contrary that it is not a weakly quasi $\varepsilon$-Pareto solution for (MD). Then, there exists $(y, \mu, \lambda, \alpha, \beta, \gamma) \in F_{MD}$ such that

$$f_i(y) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) > f_i(\bar{y}) + \sum_{\nu\in\Omega} \bar{\lambda}_\nu g_\nu(\bar{y}) + \sqrt{\theta}\|y - \bar{y}\| + \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1, \quad \forall i \in M.$$

Hence,

$$\max_{i\in M}(f_i(y)) + \sum_{\nu\in\Omega} \lambda_\nu g_\nu(y) > \max_{i\in M}(f_i(\bar{y})) + \sum_{\nu\in\Omega} \bar{\lambda}_\nu g_\nu(\bar{y}) + \sqrt{\theta}\|y - \bar{y}\| + \sqrt{\theta}\|\lambda - \bar{\lambda}\|_1,$$

Note that, by (3.17), $h_i(x) := \xi_i(f_i(x) - y_i^U), i = 1, \ldots, m$. It is easy to check that

$$H(y) + \sum_{\nu\in\Omega} \zeta_\nu g_\nu(y) > H(\bar{y}) + \sum_{\nu\in\Omega} \bar{\zeta}_\nu g_\nu(\bar{y}) + \sqrt{\theta}\|y - \bar{y}\| + \sqrt{\theta}\|\zeta - \bar{\zeta}\|_1,$$

a contradiction to (4.33). Hence, $(z, \bar{\lambda}, \bar{\mu}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is a weakly quasi $\varepsilon$-Pareto solution for (MD). □

## 5. Conclusions

This paper examines approximate Pareto solutions for multi-objective optimization problems with infinitely many constraints, employing max-ordering and weighted compromising methods. Approximate optimality conditions are established via the new concepts of $\varepsilon$-quasi subdifferentials for locally Lipschitz functions and $\varepsilon$-quasi normal sets.

The main contribution of this research lies in the development of both scalarizing and nonscalarizing methods for investigating approximate and almost-approximate solutions in multi-objective optimization. This research also extends these methods to explore the

relationship between a multi-objective optimization problem and its corresponding dual, formulated via an associated pair of single-objective problems. This duality-based approach offers a novel perspective in multi-objective optimization.

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