

Subgradient-extragradient algorithms with double inertial terms for solving image restoration problems

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ABSTRACT. In this study, we develop a class of subgradient-extragradient schemes enhanced with dual inertial mechanisms, aimed at resolving variational inequalities and fixed point formulations, particularly those arising in image reconstruction. By incorporating two inertial correction terms, the proposed algorithms aim to accelerate convergence while maintaining stability and robustness. We rigorously establish strong convergence results under suitable assumptions, extending existing theoretical frameworks to accommodate the added inertial dynamics. Numerical experiments are conducted on optimal control problems and image restoration tasks to demonstrate the practical efficiency and effectiveness of the proposed methods.

1. INTRODUCTION

Variational inequality problems (VIPs) and fixed-point problems (FPPs) have emerged as fundamental mathematical models for a wide range of applications in engineering, economics, optimal control, and image processing. These models provide a unified framework for describing equilibrium phenomena, constrained optimization problems, and inverse problems. However, the generality and flexibility of VIPs and FPPs often lead to significant analytical and computational challenges, particularly in the presence of weakly pseudomonotone operators and large-scale feasible sets.

A promising approach for addressing these problems involves projection-based iterative algorithms, including the well-known subgradient-extragradient method [25]. Such methods iteratively refine approximations by employing projections onto feasible sets together with correction steps based on operator evaluations, thereby effectively handling both operator monotonicity and feasibility constraints simultaneously. Nevertheless, classical subgradient-extragradient methods may exhibit slow convergence in practice, especially when the problem data are ill-conditioned or the underlying operators are only weakly pseudomonotone.

Recent advances have demonstrated that incorporating inertial and extrapolation terms inspired by the heavy ball method [22] and momentum-based approaches [20] can significantly accelerate convergence and improve the stability of projection-type algorithms. These inertial techniques incorporate historical information from previous iterates, thereby mitigating the limitations of purely local updates and reducing oscillatory behavior. Consequently, hybrid algorithms combining extragradient steps with inertial effects have attracted considerable attention in both theoretical and applied research.

Several studies have investigated extensions of the classical subgradient-extragradient method through the incorporation of single inertial or extrapolation terms [19, 1, 15]. Nevertheless, the performance of inertial schemes can be further improved by introducing double inertial terms, in which both the current and previous iterates contribute to the extrapolation process. Such a double-inertial mechanism has been shown to provide a more

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effective balance between stability and acceleration, particularly in large-scale or highly nonlinear settings.

Recent research has witnessed growing interest in advanced inertial-type and projection-based algorithms. Notable developments include inertial forward-backward and forward-backward-forward algorithms for variational inequality problems and monotone inclusions [4, 3], as well as variants equipped with adaptive step-size strategies to handle unknown Lipschitz constants and problem-dependent parameters [16, 13]. Double-inertial methods specifically developed for variational inequality problems involving pseudomonotone operators have been proposed in [9, 27]. Moreover, applications to image restoration and optimal control, where inertial effects are employed to suppress noise and improve image quality, have been extensively investigated in [7, 8]. In particular, inertial subgradient-extragradient schemes have demonstrated strong empirical performance on challenging imaging and inverse problems due to their capability to handle composite structures and data-driven constraints effectively [21, 10]. Nevertheless, establishing rigorous convergence guarantees for such algorithms, especially in the presence of weak pseudomonotonicity and multiple inertial corrections, remains an active area of research.

In this paper, we propose novel double-inertial subgradient-extragradient algorithms for solving variational inequality and fixed-point problems in real Hilbert spaces. The proposed algorithms incorporate two levels of inertial extrapolation into the classical extragradient framework, resulting in a unified iterative scheme with improved convergence behavior and enhanced robustness in practical applications. The main contributions of this work are summarized as follows. First, we introduce three algorithmic variants incorporating double-inertial extrapolation strategies for simultaneously solving variational inequality and fixed-point problems. Second, the proposed algorithms extend classical subgradient-extragradient methods to broader classes of operators, including weakly pseudomonotone and L -Lipschitz continuous mappings. Third, we establish rigorous convergence results under mild assumptions, such as sequential weak continuity and weak pseudomonotonicity, proving strong convergence to a common solution of the variational inequality and fixed-point problems. Finally, we present comprehensive numerical experiments on benchmark problems arising in optimal control and image restoration. The numerical results demonstrate that the proposed double-inertial subgradient-extragradient algorithms outperform existing methods in terms of both iteration count and overall computational efficiency.

The remainder of the paper is organized as follows. Section 2 presents the mathematical preliminaries, fundamental properties of weakly pseudomonotone mappings, and relevant results on projection operators. Section 3 introduces the proposed algorithms and establishes their convergence properties through a series of lemmas and theorems. Section 4 illustrates the practical performance of Algorithm 1 on benchmark numerical examples, highlighting its effectiveness and computational efficiency. Section 5 is devoted to applications of Algorithm 1 to image restoration problems. Finally, Section 6 concludes the paper and discusses possible directions for future research, including extensions to stochastic settings and distributed optimization problems.

2. PRELIMINARIES

The weak convergence of a sequence $\{x_n\}$ to \hat{x} is denoted by $x_n \rightharpoonup \hat{x}$, while its strong convergence is denoted by $x_n \rightarrow \hat{x}$ as $n \rightarrow \infty$. We now recall the following definitions and lemmas. Let $T : X \rightarrow X$ be an operator. Then T is said to be:

(m_1) A contraction, if there exists a constant $\eta \in [0, 1)$ such that

$$\|Tx - Ty\| \leq \eta \|x - y\|, \quad \forall x, y \in X.$$

(m_2) L -Lipschitz continuous, if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in X.$$

If $L = 1$, then T becomes a nonexpansive mapping.

(m_3) Quasi-nonexpansive, if $\text{Fix}(T) \neq \emptyset$ and

$$\|Tx - \hat{x}\| \leq \|x - \hat{x}\|, \quad \forall x \in X, \hat{x} \in \text{Fix}(T).$$

(m_4) α -strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in X.$$

(m_5) Monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in X.$$

(m_6) Pseudomonotone, if

$$\langle Tx, y - x \rangle \geq 0 \implies \langle Ty, y - x \rangle \geq 0, \quad \forall x, y \in X.$$

(m_7) κ -demicontractive, for some $0 \leq \kappa < 1$, if

$$\|Tx - \hat{x}\|^2 \leq \|x - \hat{x}\|^2 + \kappa\|(I - T)x\|^2, \quad \forall x \in K, \hat{x} \in \text{Fix}(T).$$

(m_8) Sequentially weakly continuous, if for any sequence $\{x_n\}$ which converges weakly to \hat{x} , the sequence $\{Tx_n\}$ weakly converges to $T\hat{x}$.

Definition 2.1. [29] Let K be a nonempty closed convex subset of a real Hilbert space X . An operator $T : K \rightarrow X$ is said to be weakly pseudomonotone if, for every sequence $\{x_n\} \subset K$ with $x_n \rightharpoonup x \in K$ and

$$\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - x \rangle \leq 0,$$

one has

$$\langle Tx, x - y \rangle \leq \liminf_{n \rightarrow \infty} \langle Tx_n, x_n - y \rangle, \quad \forall y \in K.$$

Lemma 2.1. [11] Let X be a real Hilbert space and K a nonempty closed convex subset of X . Suppose $x \in X$ and $y \in K$. Then

$$y = P_K(x) \iff \langle x - y, y - r \rangle \geq 0, \quad \forall r \in K.$$

Lemma 2.2. [11] Let K be a closed convex subset of a real Hilbert space X . If $x \in X$, then

- (i) $\|P_K(x) - P_K(y)\|^2 \leq \langle P_K(x) - P_K(y), x - y \rangle, \quad \forall x, y \in X;$
- (ii) $\|(I - P_K)x - (I - P_K)y\|^2 \leq \langle (I - P_K)x - (I - P_K)y, x - y \rangle, \quad \forall x, y \in X;$
- (iii) $\|P_K(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_K(x)\|^2, \quad \forall x \in X, y \in K.$

Lemma 2.3. [5] For each $x, y, r \in X$ and where $\alpha, \beta, \delta \in [0, 1]$ with $\alpha + \beta + \delta = 1$, the following hold in Hilbert spaces:

- (i) $\|x + y\| \leq \|x\| + \|y\|;$
- (ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle x, y \rangle + \|y\|^2;$
- (iii) $\|(1 - \alpha)x + \alpha y\|^2 = (1 - \alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$

Lemma 2.4. [11] Let $T : X \rightarrow X$ be a nonlinear operator such that $\text{Fix}(T) \neq \emptyset$. Then $I - T$ is called demiclosed at zero if for any $x_n \in X$, the following implication holds:

$$x_n \rightharpoonup \hat{x} \text{ and } (I - T)x_n \rightarrow 0 \implies \hat{x} \in \text{Fix}(T).$$

Lemma 2.5. [17] Let $T : K \rightarrow K$ be a κ -demicontractive mapping. If we define the averaged operator $T_\lambda = (1 - \lambda)I + \lambda T$, then for any $\lambda \in (0, 1 - \kappa)$, the operator T_λ is quasi-nonexpansive and satisfies $\text{Fix}(T_\lambda) = \text{Fix}(T)$.

Lemma 2.6. [28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - c_n)a_n + c_n e_n, \quad \forall n \geq 1,$$

where $\{c_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} c_n = \infty$. Assume that for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k \rightarrow \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0,$$

one has

$$\limsup_{k \rightarrow \infty} e_{n_k} \leq 0.$$

Then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

3. MAIN RESULTS

In what follows, we introduce our new types of double-inertial subgradient-extragradient algorithm for VIPs and FPPs. In order to obtain our main results, we make the following hypotheses:

- (C₁) The feasible set K is nonempty, closed, and convex.
- (C₂) The mapping $T : X \rightarrow X$ is L -Lipschitz continuous on X and weakly pseudomonotone on K .
- (C₃) The set $\Omega := \text{Fix}(F) \cap \text{VI}(K, T)$ is nonempty, closed, and convex.
- (C₄) The mapping $T : X \rightarrow X$ is sequentially weakly continuous.
- (C₅) The mapping $J : X \rightarrow X$ is nonexpansive.
- (C₆) The mapping $F : X \rightarrow X$ is κ -demicontractive for some $\kappa \in [0, 1)$, and $I - F$ is demiclosed at zero.
- (C₇) Let $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\lim_{n \rightarrow \infty} \frac{\phi_n}{\alpha_n} = 0 = \lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n}$, where $\{\phi_n\}$ and $\{\theta_n\}$ are positive real sequences.
- (C₈) The mapping $f : X \rightarrow X$ is a contraction with constant $\eta \in [0, 1)$.
- (C₉) Let $\{\gamma_n\}, \{l_n\} \subset [0, \infty)$, $\{b_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $l_n \rightarrow 0$, $b_n \rightarrow 1$, $\delta_1 > 0$.

Algorithm 1

Initialization: Choose $\varepsilon > 0, \varsigma > 0, \delta_1 > 0, \rho \in (0, 2), \mu \in (0, 1)$, and let $x_0, x_1 \in X$ arbitrarily.

Let $\{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset (0, 1), \{\gamma_n\} \subset [0, \infty), \{l_n\} \subset [0, \infty), \{b_n\} \subset [1, \infty), \{\phi_n\}$ and $\{\theta_n\}$ are positive real sequences.

Let $\{\sigma_n\}$ be a sequence satisfying $0 < a \leq \sigma_n \leq b < 1 - \kappa$ for some constants $a, b > 0$.

Iterative Procedure: Given the previous iterates x_{n-1} and x_n for $n \geq 1$, compute the next iterate x_{n+1} according to the following steps:

Step 1: Select parameters ε_n and ς_n such that $\varepsilon_n \in [0, \bar{\varepsilon}_n]$ and $\varsigma_n \in [0, \bar{\varsigma}_n]$, where

$$(3.1) \quad \bar{\varepsilon}_n = \min \left\{ \frac{n-1}{n+\varepsilon-1}, \frac{\phi_n}{\|x_n - x_{n-1}\|} \right\}, \quad \text{if } x_n \neq x_{n-1}, \quad \text{otherwise } \frac{n-1}{n+\varepsilon-1}.$$

$$(3.2) \quad \bar{\varsigma}_n = \min \left\{ \frac{n-1}{n+\varsigma-1}, \frac{\theta_n}{\|x_n - x_{n-1}\|} \right\}, \quad \text{if } x_n \neq x_{n-1}, \quad \text{otherwise } \frac{n-1}{n+\varsigma-1}.$$

Step 2: Compute

$$(3.3) \quad p_n = x_n + (1 - \alpha_n)\varepsilon_n(Jx_n - Jx_{n-1}),$$

$$(3.4) \quad q_n = x_n + (1 - \beta_n)\varsigma_n(Jx_n - Jx_{n-1}),$$

and compute

$$(3.5) \quad v_n = P_K(p_n - \delta_n T p_n).$$

If $p_n = v_n$ or $T p_n = 0$ and $p_n \in K$, then terminate; p_n is a solution of the VIP. Otherwise, proceed to Step 3.

Step 3: Compute

$$s_n = P_{M_n}(p_n - \rho \vartheta_n k_n),$$

where

$$M_n = \{z \in X : \langle p_n - \delta_n T p_n - v_n, z - v_n \rangle \leq 0\},$$

$$\vartheta_n = \begin{cases} \frac{\langle p_n - v_n, k_n \rangle}{\|k_n\|^2}, & \text{if } k_n \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$k_n = p_n - v_n - \delta_n(T p_n - T v_n).$$

Step 4: Compute

$$(3.6) \quad x_{n+1} = \alpha_n f(q_n) + (1 - \alpha_n)m_n,$$

where

$$m_n = (1 - \sigma_n)s_n + \sigma_n F s_n.$$

Update

$$(3.7) \quad \delta_{n+1} = \begin{cases} \min \left\{ \frac{(l_n + b_n \mu) \|p_n - v_n\|}{\|T p_n - T v_n\|}, \delta_n + \gamma_n \right\}, & \text{if } T p_n \neq T v_n, \\ \delta_n + \gamma_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go back to Step 1.

Remark 3.1. We make the following observation regarding Algorithm 1. It follows directly from (3.1), (3.2), and condition (C_7) that

$$(i) \quad \lim_{n \rightarrow \infty} \varepsilon_n \|x_n - x_{n-1}\| = \lim_{n \rightarrow \infty} \varsigma_n \|x_n - x_{n-1}\| = 0.$$

$$(ii) \quad \lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\alpha_n} \|x_n - x_{n-1}\| = \lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.$$

Lemma 3.7. Assume that conditions $(C_1) - (C_4)$ and (C_9) hold. Let $\{\delta_n\}$ be generated by Algorithm 1. Then the sequence $\{\delta_n\}$ is well-defined and converges to some

$$\delta \in \left[\min \left\{ \delta_1, \frac{\mu}{L} \right\}, \delta_1 + \sum_{n=1}^{\infty} \gamma_n \right].$$

Proof. We first show that $\{\delta_n\}$ is bounded below by a positive constant. If $T p_n \neq T v_n$, then by the L -Lipschitz continuity of T ,

$$\|T p_n - T v_n\| \leq L \|p_n - v_n\|.$$

Hence,

$$\frac{(l_n + b_n \mu) \|p_n - v_n\|}{\|T p_n - T v_n\|} \geq \frac{l_n + b_n \mu}{L} \geq \frac{\mu}{L},$$

since $l_n \geq 0$ and $b_n \geq 1$.

From the update rule,

$$\delta_{n+1} = \min \left\{ \frac{(l_n + b_n \mu) \|p_n - v_n\|}{\|Tp_n - Tv_n\|}, \delta_n + \gamma_n \right\},$$

we obtain

$$\delta_{n+1} \geq \min \left\{ \frac{\mu}{L}, \delta_n + \gamma_n \right\} \geq \min \left\{ \frac{\mu}{L}, \delta_n \right\}.$$

If $Tp_n = Tv_n$, then

$$\delta_{n+1} = \delta_n + \gamma_n \geq \delta_n.$$

Therefore, by induction,

$$\delta_n \geq \min \left\{ \delta_1, \frac{\mu}{L} \right\} > 0, \quad \forall n \geq 1.$$

Thus $\{\delta_n\}$ is well-defined.

Next, from the update rule, in both cases we have

$$\delta_{n+1} \leq \delta_n + \gamma_n.$$

Iterating this inequality gives

$$\delta_{n+1} \leq \delta_1 + \sum_{i=1}^n \gamma_i \leq \delta_1 + \sum_{i=1}^{\infty} \gamma_i.$$

Hence $\{\delta_n\}$ is bounded above.

Now, for any integers $m > n$,

$$\delta_m \leq \delta_n + \sum_{i=n}^{m-1} \gamma_i.$$

Taking the supremum over $m \geq n$, we obtain

$$\sup_{m \geq n} \delta_m \leq \delta_n + \sum_{i=n}^{\infty} \gamma_i.$$

Taking $\liminf_{n \rightarrow \infty}$ on both sides and using

$$\sum_{i=n}^{\infty} \gamma_i \rightarrow 0,$$

we obtain

$$\limsup_{n \rightarrow \infty} \delta_n \leq \liminf_{n \rightarrow \infty} \delta_n.$$

Therefore,

$$\limsup_{n \rightarrow \infty} \delta_n = \liminf_{n \rightarrow \infty} \delta_n,$$

so $\{\delta_n\}$ converges to some δ .

Finally, passing to the limit in the lower and upper bounds gives

$$\delta \in \left[\min \left\{ \delta_1, \frac{\mu}{L} \right\}, \delta_1 + \sum_{n=1}^{\infty} \gamma_n \right].$$

□

Lemma 3.8. *Let K be a nonempty closed convex subset of a real Hilbert space X . Let $T : K \rightarrow X$ be weakly pseudomonotone. Suppose that $\{x_n\} \subset K$, $x_n \rightharpoonup x$, and*

$$\liminf_{n \rightarrow \infty} \langle Tx_n, z - x_n \rangle \geq 0, \quad \forall z \in K.$$

Then $x \in \text{VI}(K, T)$.

Proof. Since K is closed and convex, it is weakly closed. Hence $x \in K$. Taking $z = x$ in the assumption, we obtain

$$\liminf_{n \rightarrow \infty} \langle Tx_n, x - x_n \rangle \geq 0.$$

Equivalently, this can be written as

$$\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - x \rangle \leq 0.$$

Since T is weakly pseudomonotone, it follows that, for every $y \in K$,

$$\langle Tx, x - y \rangle \leq \liminf_{n \rightarrow \infty} \langle Tx_n, x_n - y \rangle.$$

On the other hand, by the assumption with $z = y$, we have

$$\liminf_{n \rightarrow \infty} \langle Tx_n, y - x_n \rangle \geq 0.$$

Hence,

$$\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - y \rangle \leq 0.$$

Consequently,

$$\langle Tx, x - y \rangle \leq 0.$$

Therefore,

$$\langle Tx, y - x \rangle \geq 0, \quad \forall y \in K.$$

Thus,

$$x \in \text{VI}(K, T).$$

□

Lemma 3.9. *Let $\{p_n\}$ and $\{v_n\}$ be two sequences generated by Algorithm 1. Suppose that conditions $(C_1) - (C_4)$ are satisfied. If there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} \rightharpoonup \hat{x} \in X$ and*

$$\lim_{k \rightarrow \infty} \|p_{n_k} - v_{n_k}\| = 0,$$

then $\hat{x} \in \text{VI}(K, T)$.

Proof. Fix any $z \in K$. For each k , since

$$v_{n_k} = P_K(p_{n_k} - \delta_{n_k} T p_{n_k}),$$

the projection characterization (Lemma 2.1) yields

$$\langle p_{n_k} - \delta_{n_k} T p_{n_k} - v_{n_k}, v_{n_k} - z \rangle \geq 0, \quad \forall z \in K.$$

Equivalently,

$$(3.8) \quad \frac{1}{\delta_{n_k}} \langle p_{n_k} - v_{n_k}, z - v_{n_k} \rangle + \langle T p_{n_k}, v_{n_k} - p_{n_k} \rangle \leq \langle T p_{n_k}, z - p_{n_k} \rangle, \quad \forall z \in K.$$

Step 1: Boundedness and weak convergence of v_{n_k} . Since $p_{n_k} \rightharpoonup \hat{x}$, the subsequence $\{p_{n_k}\}$ is bounded in X . Because T is L -Lipschitz (hence bounded on bounded sets), $\{T p_{n_k}\}$ is bounded. Moreover, $\|p_{n_k} - v_{n_k}\| \rightarrow 0$ implies $\{v_{n_k}\}$ is bounded as well. Finally, from $\|p_{n_k} - v_{n_k}\| \rightarrow 0$ and $p_{n_k} \rightharpoonup \hat{x}$ we obtain $v_{n_k} \rightharpoonup \hat{x}$: indeed, for any $u \in X$,

$$\langle v_{n_k}, u \rangle = \langle p_{n_k}, u \rangle + \langle v_{n_k} - p_{n_k}, u \rangle \rightarrow \langle \hat{x}, u \rangle.$$

By Lemma 3.7, there exists

$$d_0 := \min \left\{ \delta_1, \frac{\mu}{L} \right\} > 0$$

such that

$$\delta_{n_k} \geq d_0, \quad \forall k \geq 1.$$

Hence all fractions are well-defined.

Step 2: Passing to \liminf in (3.8). In (3.8), the first left-hand term satisfies

$$\left| \frac{1}{\delta_{n_k}} \langle p_{n_k} - v_{n_k}, z - v_{n_k} \rangle \right| \leq \frac{1}{d_0} \|p_{n_k} - v_{n_k}\| \|z - v_{n_k}\| \rightarrow 0,$$

because $\|p_{n_k} - v_{n_k}\| \rightarrow 0$ and $\{v_{n_k}\}$ is bounded (hence $\|z - v_{n_k}\|$ is bounded). Similarly,

$$\left| \langle Tp_{n_k}, v_{n_k} - p_{n_k} \rangle \right| \leq \|Tp_{n_k}\| \|v_{n_k} - p_{n_k}\| \rightarrow 0,$$

since $\{Tp_{n_k}\}$ is bounded and $\|v_{n_k} - p_{n_k}\| \rightarrow 0$. Therefore, taking $\liminf_{k \rightarrow \infty}$ in (3.8) gives

$$(3.9) \quad \liminf_{k \rightarrow \infty} \langle Tp_{n_k}, z - p_{n_k} \rangle \geq 0, \quad \forall z \in K.$$

Step 3: Transfer of the inequality from p_{n_k} to v_{n_k} . For any $z \in K$, expand

$$(3.10) \quad \langle Tv_{n_k}, z - v_{n_k} \rangle = \langle Tv_{n_k} - Tp_{n_k}, z - p_{n_k} \rangle + \langle Tp_{n_k}, z - p_{n_k} \rangle + \langle Tv_{n_k}, p_{n_k} - v_{n_k} \rangle.$$

Since T is L -Lipschitz and $\|v_{n_k} - p_{n_k}\| \rightarrow 0$,

$$(3.11) \quad \|Tv_{n_k} - Tp_{n_k}\| \leq L \|v_{n_k} - p_{n_k}\| \rightarrow 0.$$

Moreover, $\{z - p_{n_k}\}$ is bounded (because $\{p_{n_k}\}$ is bounded), hence

$$\langle Tv_{n_k} - Tp_{n_k}, z - p_{n_k} \rangle \rightarrow 0.$$

Also $\{Tv_{n_k}\}$ is bounded (since $\{v_{n_k}\}$ is bounded and T is Lipschitz), so

$$\langle Tv_{n_k}, p_{n_k} - v_{n_k} \rangle \rightarrow 0.$$

Combining these facts with (3.9) in (3.10) yields

$$(3.12) \quad \liminf_{k \rightarrow \infty} \langle Tv_{n_k}, z - v_{n_k} \rangle \geq 0, \quad \forall z \in K.$$

Step 4: Identification of the weak limit point. Since $v_{n_k} \in K$ for all k , $v_{n_k} \rightharpoonup \hat{x}$, and K is closed and convex, it follows that

$$\hat{x} \in K.$$

From (3.12), we have

$$\liminf_{k \rightarrow \infty} \langle Tv_{n_k}, z - v_{n_k} \rangle \geq 0, \quad \forall z \in K.$$

By Lemma 3.8, applied to $x_k = v_{n_k}$, we obtain

$$\hat{x} \in \text{VI}(K, T).$$

□

Lemma 3.10. *Let conditions (C_1) – (C_4) and (C_9) be satisfied, and let the sequence $\{s_n\}$ be generated by Algorithm 1. Then, for every $\hat{x} \in \text{VI}(K, T) \cap \text{Fix}(F)$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,*

$$(3.13) \quad \begin{aligned} \|s_n - \hat{x}\|^2 &\leq \|p_n - \hat{x}\|^2 - \|p_n - s_n - \rho\vartheta_n k_n\|^2 \\ &\quad - (2 - \rho)\rho \left(\frac{1 - \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}} \right)^2 \|p_n - v_n\|^2. \end{aligned}$$

Proof. Let $\hat{x} \in \text{VI}(K, T) \cap \text{Fix}(F)$, and define

$$a_n := p_n - v_n \quad \text{and} \quad d_n := Tp_n - Tv_n.$$

Then

$$k_n = a_n - \delta_n d_n.$$

Since

$$v_n = P_K(p_n - \delta_n Tp_n),$$

the characterization of the metric projection yields

$$\langle p_n - \delta_n Tp_n - v_n, z - v_n \rangle \leq 0, \quad \forall z \in K.$$

Hence, the half-space

$$M_n := \{z \in X : \langle p_n - \delta_n Tp_n - v_n, z - v_n \rangle \leq 0\}$$

contains K . Since $\hat{x} \in K$, we obtain

$$\hat{x} \in M_n.$$

Suppose first that $Tp_n \neq Tv_n$. By the step-size rule in Algorithm 1,

$$\delta_{n+1} = \min \left\{ \frac{(l_n + b_n \mu) \|p_n - v_n\|}{\|Tp_n - Tv_n\|}, \delta_n + \gamma_n \right\}.$$

Therefore,

$$\delta_{n+1} \leq \frac{(l_n + b_n \mu) \|p_n - v_n\|}{\|Tp_n - Tv_n\|}.$$

Consequently,

$$\|d_n\| = \|Tp_n - Tv_n\| \leq \frac{l_n + b_n \mu}{\delta_{n+1}} \|a_n\|.$$

If $Tp_n = Tv_n$, then $d_n = 0$, and the above inequality holds trivially.

Hence,

$$\begin{aligned} \|k_n\| &= \|a_n - \delta_n d_n\| \\ &\leq \|a_n\| + \delta_n \|d_n\| \\ &\leq \left(1 + \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}} \right) \|a_n\|. \end{aligned}$$

Moreover,

$$\begin{aligned} \langle a_n, k_n \rangle &= \langle a_n, a_n - \delta_n d_n \rangle \\ &= \|a_n\|^2 - \delta_n \langle a_n, d_n \rangle \\ &\geq \|a_n\|^2 - \delta_n \|a_n\| \|d_n\| \\ &\geq \left(1 - \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}} \right) \|a_n\|^2. \end{aligned}$$

By Lemma 3.7, $\delta_n \rightarrow \delta > 0$. Hence,

$$\frac{\delta_n}{\delta_{n+1}} \rightarrow 1.$$

Since $l_n \rightarrow 0$ and $b_n \rightarrow 1$, we obtain

$$\frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}} \rightarrow \mu < 1.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$1 - \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}} > 0.$$

Since

$$s_n = P_{M_n}(p_n - \rho\vartheta_n k_n)$$

and $\hat{x} \in M_n$, Lemma 2.2(iii) implies that

$$\|s_n - \hat{x}\|^2 \leq \|p_n - \rho\vartheta_n k_n - \hat{x}\|^2 - \|p_n - \rho\vartheta_n k_n - s_n\|^2.$$

Expanding the first term, we obtain

$$\begin{aligned} \|s_n - \hat{x}\|^2 &\leq \|p_n - \hat{x}\|^2 - \|p_n - s_n - \rho\vartheta_n k_n\|^2 \\ &\quad - 2\rho\vartheta_n \langle k_n, s_n - \hat{x} \rangle. \end{aligned}$$

Since $s_n = P_{M_n}(p_n - \rho\vartheta_n k_n)$ and $\hat{x} \in M_n$, the projection property yields

$$\langle p_n - \rho\vartheta_n k_n - s_n, \hat{x} - s_n \rangle \leq 0.$$

Equivalently,

$$\langle p_n - s_n, \hat{x} - s_n \rangle \leq \rho\vartheta_n \langle k_n, \hat{x} - s_n \rangle.$$

Using the identity

$$2\langle p_n - s_n, \hat{x} - s_n \rangle = \|p_n - \hat{x}\|^2 - \|p_n - s_n\|^2 - \|s_n - \hat{x}\|^2,$$

together with the previous inequality, we obtain

$$\|s_n - \hat{x}\|^2 \leq \|p_n - \hat{x}\|^2 - \|p_n - s_n - \rho\vartheta_n k_n\|^2 - (2 - \rho)\rho\vartheta_n^2 \|k_n\|^2.$$

Finally, since

$$\vartheta_n = \frac{\langle a_n, k_n \rangle}{\|k_n\|^2},$$

we have

$$\vartheta_n^2 \|k_n\|^2 = \frac{\langle a_n, k_n \rangle^2}{\|k_n\|^2}.$$

Using the lower bound for $\langle a_n, k_n \rangle$ and the upper bound for $\|k_n\|$, we obtain

$$\begin{aligned} \vartheta_n^2 \|k_n\|^2 &\geq \frac{\left(1 - \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}\right)^2 \|a_n\|^4}{\left(1 + \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}\right)^2 \|a_n\|^2} \\ &= \left(\frac{1 - \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}}\right)^2 \|a_n\|^2. \end{aligned}$$

Since $a_n = p_n - v_n$, the desired inequality follows. □

Theorem 3.1. *Assume that conditions (C₁) – (C₉) hold, and let {x_n} be the sequence generated by Algorithm 1. Then the sequence {x_n} converges strongly to the unique point $\hat{x} \in \Omega := \text{Fix}(F) \cap \text{VI}(K, T)$ satisfying $\hat{x} = P_\Omega(f(\hat{x}))$.*

Proof. We divide the proof into four parts as follows:

Claim 1. The sequence {x_n} is bounded.

First, we consider the definition of the intermediate sequence m_n from Algorithm 1, which is given by

$$m_n = (1 - \sigma_n)s_n + \sigma_n F s_n.$$

Using the properties of the norm in a Hilbert space together with the κ -demicontractivity of F in condition (C_6) , we obtain

$$\begin{aligned} \|m_n - \hat{x}\|^2 &= \|(1 - \sigma_n)s_n + \sigma_n F s_n - \hat{x}\|^2 \\ &= \|(s_n - \hat{x}) + \sigma_n(F s_n - s_n)\|^2 \\ &= \|s_n - \hat{x}\|^2 + 2\sigma_n \langle s_n - \hat{x}, F s_n - s_n \rangle + \sigma_n^2 \|F s_n - s_n\|^2 \\ &\leq \|s_n - \hat{x}\|^2 + \sigma_n(\kappa - 1) \|F s_n - s_n\|^2 + \sigma_n^2 \|F s_n - s_n\|^2 \\ &= \|s_n - \hat{x}\|^2 - \sigma_n(1 - \kappa - \sigma_n) \|F s_n - s_n\|^2. \end{aligned}$$

By applying the estimate for $\|s_n - \hat{x}\|^2$ from Lemma 3.10 and using the assumption $0 < a \leq \sigma_n \leq b < 1 - \kappa$, we obtain

(3.14)

$$\begin{aligned} \|m_n - \hat{x}\|^2 &\leq \|p_n - \hat{x}\|^2 - \|p_n - s_n - \rho \vartheta_n k_n\|^2 - (2 - \rho)\rho \left(\frac{1 - \frac{(l_n + b_n \mu)\delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n \mu)\delta_n}{\delta_{n+1}}} \right)^2 \|p_n - v_n\|^2 \\ &\quad - \sigma_n(1 - \kappa - \sigma_n) \|F s_n - s_n\|^2. \end{aligned}$$

By condition (C_9) and Lemma 3.7, we have $l_n \rightarrow 0, b_n \rightarrow 1$, and $\delta_n \rightarrow \delta > 0$. Hence

$$\frac{(l_n + b_n \mu)\delta_n}{\delta_{n+1}} \rightarrow \mu.$$

Consequently, there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the coefficient

$$1 - \frac{(l_n + b_n \mu)\delta_n}{\delta_{n+1}} > 0.$$

Substituting this into (3.14), we conclude that for all $n \geq n_0$, all subtracted terms on the right-hand side are non-negative, which yields

$$(3.15) \quad \|m_n - \hat{x}\|^2 \leq \|p_n - \hat{x}\|^2 \implies \|m_n - \hat{x}\| \leq \|p_n - \hat{x}\|.$$

Next, we analyze the boundedness of the inertial terms. By the definition of p_n in Algorithm 1 and condition (C_5) , we have

$$\begin{aligned} \|p_n - \hat{x}\| &= \|x_n + (1 - \alpha_n)\varepsilon_n(Jx_n - Jx_{n-1}) - \hat{x}\| \\ &= \|(x_n - \hat{x}) + (1 - \alpha_n)\varepsilon_n(Jx_n - Jx_{n-1})\| \\ (3.16) \quad &\leq \|x_n - \hat{x}\| + (1 - \alpha_n)\varepsilon_n \|Jx_n - Jx_{n-1}\| \\ &\leq \|x_n - \hat{x}\| + (1 - \alpha_n)\varepsilon_n \|x_n - x_{n-1}\| \\ &= \|x_n - \hat{x}\| + \alpha_n \left[(1 - \alpha_n) \frac{\varepsilon_n}{\alpha_n} \|x_n - x_{n-1}\| \right]. \end{aligned}$$

According to Remark 3.1 and condition (C_7) , the term $(1 - \alpha_n) \frac{\varepsilon_n}{\alpha_n} \|x_n - x_{n-1}\|$ converges to 0, and thus it is bounded by some constant $\Upsilon_1 > 0$. Therefore,

$$\|p_n - \hat{x}\| \leq \|x_n - \hat{x}\| + \alpha_n \Upsilon_1.$$

By the definition of q_n in Algorithm 1 and condition (C_5) , we have

$$\begin{aligned}
 \|q_n - \hat{x}\| &= \|x_n + (1 - \beta_n)\varsigma_n(Jx_n - Jx_{n-1}) - \hat{x}\| \\
 &= \|(x_n - \hat{x}) + (1 - \beta_n)\varsigma_n(Jx_n - Jx_{n-1})\| \\
 (3.17) \quad &\leq \|x_n - \hat{x}\| + (1 - \beta_n)\varsigma_n\|Jx_n - Jx_{n-1}\| \\
 &\leq \|x_n - \hat{x}\| + (1 - \beta_n)\varsigma_n\|x_n - x_{n-1}\| \\
 &= \|x_n - \hat{x}\| + \alpha_n \left[(1 - \beta_n) \frac{\varsigma_n}{\alpha_n} \|x_n - x_{n-1}\| \right].
 \end{aligned}$$

According to Remark 3.1 and condition (C_7) , the term $(1 - \beta_n) \frac{\varsigma_n}{\alpha_n} \|x_n - x_{n-1}\|$ converges to 0, and thus it is bounded by some constant $\Upsilon_2 > 0$. Therefore,

$$\|q_n - \hat{x}\| \leq \|x_n - \hat{x}\| + \alpha_n \Upsilon_2.$$

By the definition of x_{n+1} in Algorithm 1 and condition (C_8) , we have

$$\begin{aligned}
 \|x_{n+1} - \hat{x}\| &= \|\alpha_n f(q_n) + (1 - \alpha_n)m_n - \hat{x}\| \\
 (3.18) \quad &\leq \alpha_n \|f(q_n) - f(\hat{x})\| + \alpha_n \|f(\hat{x}) - \hat{x}\| + (1 - \alpha_n) \|m_n - \hat{x}\| \\
 &\leq \alpha_n \eta \|q_n - \hat{x}\| + \alpha_n \|f(\hat{x}) - \hat{x}\| + (1 - \alpha_n) \|p_n - \hat{x}\|.
 \end{aligned}$$

Substituting (3.16) and (3.17) into (3.18), we have

$$\begin{aligned}
 \|x_{n+1} - \hat{x}\| &\leq \alpha_n \eta (\|x_n - \hat{x}\| + \alpha_n \Upsilon_2) + \alpha_n \|f(\hat{x}) - \hat{x}\| + (1 - \alpha_n) (\|x_n - \hat{x}\| + \alpha_n \Upsilon_1) \\
 &\leq (1 - \alpha_n (1 - \eta)) \|x_n - \hat{x}\| + \alpha_n (\eta \Upsilon_2 + \Upsilon_1 + \|f(\hat{x}) - \hat{x}\|) \\
 &= (1 - \alpha_n (1 - \eta)) \|x_n - \hat{x}\| + \alpha_n (1 - \eta) \left(\frac{\|f(\hat{x}) - \hat{x}\| + \Upsilon_3}{1 - \eta} \right),
 \end{aligned}$$

where $\Upsilon_3 = \eta \Upsilon_2 + \Upsilon_1$. By iterating the above recursive inequality, we obtain

$$\|x_n - \hat{x}\| \leq \max \left\{ \|x_{n_0} - \hat{x}\|, \frac{\|f(\hat{x}) - \hat{x}\| + \Upsilon_3}{1 - \eta} \right\}, \quad \forall n \geq n_0.$$

Hence, the sequence $\{x_n\}$ is bounded, which further implies that $\{p_n\}$, $\{q_n\}$, $\{s_n\}$, and $\{m_n\}$ are also bounded.

Claim 2. The following inequality holds for all $n \geq n_0$:

$$\begin{aligned}
 (1 - \alpha_n) \|p_n - s_n - \rho \vartheta_n k_n\|^2 + (1 - \alpha_n) (2 - \rho) \rho \left(\frac{1 - \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}} \right)^2 \|p_n - v_n\|^2 \\
 + (1 - \alpha_n) \sigma_n (1 - \kappa - \sigma_n) \|s_n - F s_n\|^2 \leq \|x_n - \hat{x}\|^2 - \|x_{n+1} - \hat{x}\|^2 + \alpha_n \Upsilon_7,
 \end{aligned}$$

for some constant $\Upsilon_7 > 0$.

From Lemma 3.10, we obtain

$$(3.19) \quad \|s_n - \hat{x}\| \leq \|p_n - \hat{x}\|.$$

By using (3.16) and (3.19), we have the following estimate:

$$(3.20) \quad \|s_n - \hat{x}\| \leq \|p_n - \hat{x}\| \leq \|x_n - \hat{x}\| + \alpha_n \Upsilon_1.$$

Squaring both sides of the inequality for $\|p_n - \hat{x}\|$, we get

$$\begin{aligned}
 (3.21) \quad \|p_n - \hat{x}\|^2 &\leq (\|x_n - \hat{x}\| + \alpha_n \Upsilon_1)^2 \\
 &= \|x_n - \hat{x}\|^2 + \alpha_n (2\Upsilon_1 \|x_n - \hat{x}\| + \alpha_n \Upsilon_1^2).
 \end{aligned}$$

Since the sequence $\{x_n\}$ is bounded, it implies that the term in the parenthesis is also bounded. Thus, there exists a constant $\Upsilon_4 > 0$ such that $2\Upsilon_1\|x_n - \hat{x}\| + \alpha_n\Upsilon_1^2 \leq \Upsilon_4$ for all $n \in \mathbb{N}$. Therefore, inequality (3.21) can be rewritten as

$$\|p_n - \hat{x}\|^2 \leq \|x_n - \hat{x}\|^2 + \alpha_n\Upsilon_4.$$

Similarly, from the definition of q_n in (3.17), we obtain

$$(3.22) \quad \begin{aligned} \|q_n - \hat{x}\|^2 &\leq (\|x_n - \hat{x}\| + \alpha_n\Upsilon_2)^2 \\ &= \|x_n - \hat{x}\|^2 + \alpha_n(2\Upsilon_2\|x_n - \hat{x}\| + \alpha_n\Upsilon_2^2). \end{aligned}$$

Due to the boundedness of $\{x_n\}$, there exists a constant $\Upsilon_5 > 0$ such that $2\Upsilon_2\|x_n - \hat{x}\| + \alpha_n\Upsilon_2^2 \leq \Upsilon_5$ for all $n \in \mathbb{N}$, which leads to

$$\|q_n - \hat{x}\|^2 \leq \|x_n - \hat{x}\|^2 + \alpha_n\Upsilon_5.$$

Next, we examine the term involving the contraction mapping f :

$$\alpha_n\|f(q_n) - \hat{x}\|^2 = \alpha_n\|f(q_n) - f(\hat{x}) + f(\hat{x}) - \hat{x}\|^2.$$

Using the property $\|a + b\|^2 \leq \|a\|^2 + \|b\|^2 + 2\|a\|\|b\|$ and condition (C_8) , we deduce

$$(3.23) \quad \begin{aligned} \alpha_n\|f(q_n) - \hat{x}\|^2 &\leq \alpha_n\|f(q_n) - f(\hat{x})\|^2 + \alpha_n\|f(\hat{x}) - \hat{x}\|^2 + 2\alpha_n\|f(q_n) - f(\hat{x})\|\|f(\hat{x}) - \hat{x}\| \\ &\leq \alpha_n\eta^2\|q_n - \hat{x}\|^2 + \alpha_n\|f(\hat{x}) - \hat{x}\|^2 + 2\alpha_n\eta\|q_n - \hat{x}\|\|f(\hat{x}) - \hat{x}\| \\ &\leq \alpha_n\eta^2(\|x_n - \hat{x}\|^2 + \alpha_n\Upsilon_5) + \alpha_n\|f(\hat{x}) - \hat{x}\|^2 + 2\alpha_n\eta\|q_n - \hat{x}\|\|f(\hat{x}) - \hat{x}\| \\ &\leq \alpha_n\|x_n - \hat{x}\|^2 + \alpha_n[\eta^2\alpha_n\Upsilon_5 + \|f(\hat{x}) - \hat{x}\|^2 + 2\eta\|q_n - \hat{x}\|\|f(\hat{x}) - \hat{x}\|], \end{aligned}$$

where we used the fact that $\eta^2 < 1$. Since the sequences $\{x_n\}$ and $\{q_n\}$ are bounded, there exists a constant $\Upsilon_6 > 0$ such that

$$\eta^2\alpha_n\Upsilon_5 + \|f(\hat{x}) - \hat{x}\|^2 + 2\eta\|q_n - \hat{x}\|\|f(\hat{x}) - \hat{x}\| \leq \Upsilon_6, \quad \forall n \in \mathbb{N}.$$

Therefore, inequality (3.23) simplifies to

$$\alpha_n\|f(q_n) - \hat{x}\|^2 \leq \alpha_n\|x_n - \hat{x}\|^2 + \alpha_n\Upsilon_6.$$

Finally, using the convexity of the squared norm, we obtain

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &= \|\alpha_n f(q_n) + (1 - \alpha_n)m_n - \hat{x}\|^2 \\ &\leq \alpha_n\|f(q_n) - \hat{x}\|^2 + (1 - \alpha_n)\|m_n - \hat{x}\|^2 \\ &\leq \alpha_n\|f(q_n) - \hat{x}\|^2 + (1 - \alpha_n)\|p_n - \hat{x}\|^2 \\ &\quad - (1 - \alpha_n)\|p_n - s_n - \rho\vartheta_n k_n\|^2 \\ &\quad - (1 - \alpha_n)(2 - \rho)\rho \left(\frac{1 - \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}} \right)^2 \|p_n - v_n\|^2 \\ &\quad - (1 - \alpha_n)\sigma_n(1 - \kappa - \sigma_n)\|F s_n - s_n\|^2 \end{aligned}$$

$$\begin{aligned}
& \text{Duangkamon Kitkuan, Anantachai Padcharoen and Tanakit Thianwan} \\
& \leq \alpha_n \|x_n - \hat{x}\|^2 + \alpha_n \Upsilon_6 + (1 - \alpha_n) \|x_n - \hat{x}\|^2 + \alpha_n \Upsilon_4 \\
& \quad - (1 - \alpha_n) \|p_n - s_n - \rho \vartheta_n k_n\|^2 \\
& \quad - (1 - \alpha_n)(2 - \rho) \rho \left(\frac{1 - \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}} \right)^2 \|p_n - v_n\|^2 \\
& \quad - (1 - \alpha_n) \sigma_n (1 - \kappa - \sigma_n) \|F s_n - s_n\|^2 \\
& = \|x_n - \hat{x}\|^2 + \alpha_n (\Upsilon_6 + \Upsilon_4) \\
& \quad - (1 - \alpha_n) \|p_n - s_n - \rho \vartheta_n k_n\|^2 \\
& \quad - (1 - \alpha_n)(2 - \rho) \rho \left(\frac{1 - \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}} \right)^2 \|p_n - v_n\|^2 \\
& \quad - (1 - \alpha_n) \sigma_n (1 - \kappa - \sigma_n) \|F s_n - s_n\|^2.
\end{aligned}$$

This implies the following fundamental inequality:

$$\begin{aligned}
(3.24) \quad & (1 - \alpha_n) \|p_n - s_n - \rho \vartheta_n k_n\|^2 + (1 - \alpha_n)(2 - \rho) \rho \left(\frac{1 - \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n \mu) \delta_n}{\delta_{n+1}}} \right)^2 \|p_n - v_n\|^2 \\
& + (1 - \alpha_n) \sigma_n (1 - \kappa - \sigma_n) \|F s_n - s_n\|^2 \leq \|x_n - \hat{x}\|^2 - \|x_{n+1} - \hat{x}\|^2 + \alpha_n \Upsilon_7,
\end{aligned}$$

valid for all $n \geq n_0$, where $\Upsilon_7 = \Upsilon_4 + \Upsilon_6 > 0$.

Claim 3. The following estimate holds:

$$\begin{aligned}
\|x_{n+1} - \hat{x}\|^2 & \leq (1 - (1 - \eta)\alpha_n) \|x_n - \hat{x}\|^2 + (1 - \eta)\alpha_n \left[\frac{2}{1 - \eta} \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x} \rangle \right. \\
& \quad \left. + \frac{3\Upsilon_8 \varepsilon_n}{(1 - \eta)\alpha_n} \|x_n - x_{n-1}\| + \frac{3\eta \Upsilon_9 \varsigma_n}{1 - \eta} \|x_n - x_{n-1}\| \right].
\end{aligned}$$

First, by the definition of p_n in Algorithm 1 and condition (C_5) , we have

$$\begin{aligned}
\|p_n - \hat{x}\| & = \|x_n + (1 - \alpha_n)\varepsilon_n(Jx_n - Jx_{n-1}) - \hat{x}\| \\
& \leq \|x_n - \hat{x}\| + (1 - \alpha_n)\varepsilon_n \|Jx_n - Jx_{n-1}\| \\
& \leq \|x_n - \hat{x}\| + (1 - \alpha_n)\varepsilon_n \|x_n - x_{n-1}\|.
\end{aligned}$$

By the triangle inequality and the fact that $(1 - \alpha_n) \leq 1$, it follows that

$$(3.25) \quad \|p_n - \hat{x}\| \leq \|x_n - \hat{x}\| + \varepsilon_n \|x_n - x_{n-1}\|.$$

Squaring both sides of (3.25) and using the boundedness of the sequences, we deduce

$$\begin{aligned}
(3.26) \quad & \|p_n - \hat{x}\|^2 \leq (\|x_n - \hat{x}\| + \varepsilon_n \|x_n - x_{n-1}\|)^2 \\
& = \|x_n - \hat{x}\|^2 + 2\varepsilon_n \|x_n - \hat{x}\| \|x_n - x_{n-1}\| + \varepsilon_n^2 \|x_n - x_{n-1}\|^2 \\
& \leq \|x_n - \hat{x}\|^2 + 3\varepsilon_n \|x_n - x_{n-1}\| \max\{\|x_n - \hat{x}\|, \varepsilon_n \|x_n - x_{n-1}\|\} \\
& \leq \|x_n - \hat{x}\|^2 + 3\Upsilon_8 \varepsilon_n \|x_n - x_{n-1}\|,
\end{aligned}$$

where $\Upsilon_8 = \sup_{n \in \mathbb{N}} \{\|x_n - \hat{x}\|, \varepsilon_n \|x_n - x_{n-1}\|\}$. Similarly, from the definition of q_n in Algorithm 1 and condition (C_5) , we have

$$\begin{aligned}
\|q_n - \hat{x}\| & = \|x_n + (1 - \beta_n)\varsigma_n(Jx_n - Jx_{n-1}) - \hat{x}\| \\
& \leq \|x_n - \hat{x}\| + (1 - \beta_n)\varsigma_n \|Jx_n - Jx_{n-1}\| \\
& \leq \|x_n - \hat{x}\| + (1 - \beta_n)\varsigma_n \|x_n - x_{n-1}\|.
\end{aligned}$$

By the triangle inequality and the fact that $(1 - \beta_n) \leq 1$, it follows that

$$(3.27) \quad \|q_n - \hat{x}\| \leq \|x_n - \hat{x}\| + \varsigma_n \|x_n - x_{n-1}\|.$$

Squaring both sides of (3.27) and using the boundedness of the sequences, we deduce

$$\begin{aligned} \|q_n - \hat{x}\|^2 &\leq (\|x_n - \hat{x}\| + \varsigma_n \|x_n - x_{n-1}\|)^2 \\ &= \|x_n - \hat{x}\|^2 + 2\varsigma_n \|x_n - \hat{x}\| \|x_n - x_{n-1}\| + \varsigma_n^2 \|x_n - x_{n-1}\|^2 \\ &= \|x_n - \hat{x}\|^2 + \varsigma_n \|x_n - x_{n-1}\| (2\|x_n - \hat{x}\| + \varsigma_n \|x_n - x_{n-1}\|). \end{aligned}$$

Let $\Upsilon_9 = \sup_{n \in \mathbb{N}} \{\|x_n - \hat{x}\|, \varsigma_n \|x_n - x_{n-1}\|\}$. Then, we have

$$(3.28) \quad \|q_n - \hat{x}\|^2 \leq \|x_n - \hat{x}\|^2 + 3\Upsilon_9 \varsigma_n \|x_n - x_{n-1}\|.$$

Next, using the convexity of the norm squared and the property $\|a + b\|^2 \leq \|a\|^2 + 2\langle b, a + b \rangle$, and since $\eta^2 \leq \eta$ for $0 \leq \eta < 1$, we obtain

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &= \|\alpha_n [f(q_n) - f(\hat{x})] + (1 - \alpha_n)(m_n - \hat{x}) + \alpha_n (f(\hat{x}) - \hat{x})\|^2 \\ &\leq \|\alpha_n (f(q_n) - f(\hat{x})) + (1 - \alpha_n)(m_n - \hat{x})\|^2 + 2\alpha_n \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x} \rangle \\ &\leq \alpha_n \eta \|q_n - \hat{x}\|^2 + (1 - \alpha_n) \|m_n - \hat{x}\|^2 + 2\alpha_n \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x} \rangle. \end{aligned}$$

Substituting the estimates $\|m_n - \hat{x}\| \leq \|p_n - \hat{x}\|$ from Claim 1, along with (3.26) and (3.28):

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leq \alpha_n \eta (\|x_n - \hat{x}\|^2 + 3\Upsilon_9 \varsigma_n \|x_n - x_{n-1}\|) \\ &\quad + (1 - \alpha_n) (\|x_n - \hat{x}\|^2 + 3\Upsilon_8 \varepsilon_n \|x_n - x_{n-1}\|) \\ &\quad + 2\alpha_n \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x} \rangle \\ &\leq (1 - (1 - \eta)\alpha_n) \|x_n - \hat{x}\|^2 + 3(1 - \alpha_n) \Upsilon_8 \varepsilon_n \|x_n - x_{n-1}\| \\ &\quad + 3\alpha_n \eta \Upsilon_9 \varsigma_n \|x_n - x_{n-1}\| + 2\alpha_n \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x} \rangle. \end{aligned}$$

Finally, by factoring out $(1 - \eta)\alpha_n$ and noting that $(1 - \alpha_n) \leq 1$, we conclude

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leq (1 - (1 - \eta)\alpha_n) \|x_n - \hat{x}\|^2 + (1 - \eta)\alpha_n \left[\frac{2}{1 - \eta} \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x} \rangle \right. \\ &\quad \left. + \frac{3\Upsilon_8 \varepsilon_n}{(1 - \eta)\alpha_n} \|x_n - x_{n-1}\| + \frac{3\eta \Upsilon_9 \varsigma_n}{1 - \eta} \|x_n - x_{n-1}\| \right]. \end{aligned}$$

Claim 4. The sequence $\{\|x_n - \hat{x}\|^2\}$ converges to zero.

Define

$$\begin{aligned} a_n &:= \|x_n - \hat{x}\|^2, \\ e_n &:= \frac{2}{1 - \eta} \langle f(\hat{x}) - \hat{x}, x_{n+1} - \hat{x} \rangle + \frac{3\Upsilon_8 \varepsilon_n}{(1 - \eta)\alpha_n} \|x_n - x_{n-1}\| + \frac{3\eta \Upsilon_9 \varsigma_n}{1 - \eta} \|x_n - x_{n-1}\|. \end{aligned}$$

Then Claim 3 can be restated as

$$a_{n+1} \leq (1 - c_n) a_n + c_n e_n, \quad \text{where } c_n = (1 - \eta)\alpha_n.$$

By Lemma 2.6, it suffices to show that $\limsup_{k \rightarrow \infty} e_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ such that $\liminf_{k \rightarrow \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0$. Let $\{\|x_{n_k} - \hat{x}\|\}$ be such a subsequence. Then,

$$\begin{aligned} &\liminf_{k \rightarrow \infty} \left(\|x_{n_{k+1}} - \hat{x}\|^2 - \|x_{n_k} - \hat{x}\|^2 \right) \\ &= \liminf_{k \rightarrow \infty} \left(\|x_{n_{k+1}} - \hat{x}\| - \|x_{n_k} - \hat{x}\| \right) \left(\|x_{n_{k+1}} - \hat{x}\| + \|x_{n_k} - \hat{x}\| \right) \geq 0. \end{aligned}$$

Since $\alpha_n \rightarrow 0$ and $0 < a \leq \sigma_n \leq b < 1 - \kappa$, there exists $N \in \mathbb{N}$ such that $1 - \alpha_n \geq \frac{1}{2}$ for all $n \geq N$. Hence,

$$(1 - \alpha_n)\sigma_n(1 - \kappa - \sigma_n) \geq \frac{1}{2}a(1 - \kappa - b) =: c_1 > 0.$$

Since

$$\frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}} \rightarrow \mu < 1,$$

there exists a constant $c_2 > 0$ such that

$$(1 - \alpha_n)(2 - \rho)\rho \left(\frac{1 - \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}}{1 + \frac{(l_n + b_n\mu)\delta_n}{\delta_{n+1}}} \right)^2 \geq c_2$$

for all sufficiently large n .

By Claim 2 and condition (C_7) , we get

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left[(1 - \alpha_{n_k}) \|p_{n_k} - s_{n_k} - \rho\vartheta_{n_k} k_{n_k}\|^2 \right. \\ & \quad + (1 - \alpha_{n_k})(2 - \rho)\rho \left(\frac{1 - \frac{(l_{n_k} + b_{n_k}\mu)\delta_{n_k}}{\delta_{n_k+1}}}{1 + \frac{(l_{n_k} + b_{n_k}\mu)\delta_{n_k}}{\delta_{n_k+1}}} \right)^2 \|p_{n_k} - v_{n_k}\|^2 \\ & \quad \left. + (1 - \alpha_{n_k})\sigma_{n_k}(1 - \kappa - \sigma_{n_k}) \|s_{n_k} - F s_{n_k}\|^2 \right] \\ & \leq \limsup_{k \rightarrow \infty} \left[\|x_{n_k} - \hat{x}\|^2 - \|x_{n_k+1} - \hat{x}\|^2 + \alpha_{n_k} \Upsilon_7 \right] \leq 0, \end{aligned}$$

which implies that

$$(3.29) \quad \lim_{k \rightarrow \infty} \|p_{n_k} - s_{n_k} - \rho\vartheta_{n_k} k_{n_k}\| = \lim_{k \rightarrow \infty} \|p_{n_k} - v_{n_k}\| = \lim_{k \rightarrow \infty} \|s_{n_k} - F s_{n_k}\| = 0.$$

Furthermore, by using the triangle inequality and Cauchy-Schwarz:

$$\begin{aligned} (3.30) \quad \|p_{n_k} - s_{n_k}\| & \leq \|p_{n_k} - s_{n_k} - \rho\vartheta_{n_k} k_{n_k}\| + \rho\vartheta_{n_k} \|k_{n_k}\| \\ & = \|p_{n_k} - s_{n_k} - \rho\vartheta_{n_k} k_{n_k}\| + \rho \frac{\langle p_{n_k} - v_{n_k}, k_{n_k} \rangle}{\|k_{n_k}\|} \\ & \leq \|p_{n_k} - s_{n_k} - \rho\vartheta_{n_k} k_{n_k}\| + \rho \|p_{n_k} - v_{n_k}\|. \end{aligned}$$

Recalling (3.29), it follows that $\lim_{k \rightarrow \infty} \|p_{n_k} - s_{n_k}\| = 0$. By condition (C_7) and the definitions of p_{n_k} and q_{n_k} , we have

$$(3.31) \quad \lim_{k \rightarrow \infty} \|p_{n_k} - x_{n_k}\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|q_{n_k} - x_{n_k}\| = 0.$$

Using (3.30) and (3.31), we obtain

$$(3.32) \quad \lim_{k \rightarrow \infty} \|s_{n_k} - x_{n_k}\| \leq \lim_{k \rightarrow \infty} [\|s_{n_k} - p_{n_k}\| + \|p_{n_k} - x_{n_k}\|] = 0.$$

From $m_{n_k} = (1 - \sigma_{n_k})s_{n_k} + \sigma_{n_k}F s_{n_k}$, we deduce

$$(3.33) \quad \|m_{n_k} - s_{n_k}\| = \sigma_{n_k} \|F s_{n_k} - s_{n_k}\| \leq \|F s_{n_k} - s_{n_k}\| \rightarrow 0.$$

Combining (3.33) and (3.32), and using the definition

$$x_{n+1} = \alpha_n f(q_n) + (1 - \alpha_n)m_n,$$

we estimate

$$(3.34) \quad \begin{aligned} \|x_{n_k+1} - x_{n_k}\| &\leq \alpha_{n_k} \|f(q_{n_k}) - x_{n_k}\| + (1 - \alpha_{n_k}) \|m_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k} \|f(q_{n_k}) - x_{n_k}\| + \|m_{n_k} - s_{n_k}\| + \|s_{n_k} - x_{n_k}\|. \end{aligned}$$

Since $\{x_n\}$ and $\{q_n\}$ are bounded and f is a contraction, $\|f(q_{n_k}) - x_{n_k}\|$ is bounded. Moreover, by condition (C_7) , $\alpha_{n_k} \rightarrow 0$. Together with (3.33) and (3.32), we obtain

$$(3.35) \quad \|x_{n_k+1} - x_{n_k}\| \rightarrow 0 \quad (k \rightarrow \infty).$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ such that

$$(3.36) \quad x_{n_{k_j}} \rightharpoonup \omega \quad (j \rightarrow \infty).$$

Step 1: $\omega \in \text{Fix}(F)$. From (3.32) we have $s_{n_{k_j}} - x_{n_{k_j}} \rightarrow 0$, hence $s_{n_{k_j}} \rightharpoonup \omega$. By (3.29),

$$\|s_{n_{k_j}} - F s_{n_{k_j}}\| \rightarrow 0,$$

that is,

$$(I - F)s_{n_{k_j}} \rightarrow 0.$$

Since $I - F$ is demiclosed at 0 by condition (C_6) , we conclude that

$$\omega \in \text{Fix}(F).$$

Step 2: $\omega \in \text{VI}(K, T)$. From condition (C_7) and the definition of p_n , we have

$$\|p_{n_k} - x_{n_k}\| \rightarrow 0,$$

hence $p_{n_{k_j}} \rightharpoonup \omega$. Together with (3.29), which gives

$$\|p_{n_k} - v_{n_k}\| \rightarrow 0,$$

Lemma 3.9 yields

$$\omega \in \text{VI}(K, T).$$

Therefore,

$$(3.37) \quad \omega \in \Omega := \text{Fix}(F) \cap \text{VI}(K, T).$$

Recall that $\hat{x} = P_\Omega(f(\hat{x}))$. Hence, by the characterization of the metric projection,

$$(3.38) \quad \langle f(\hat{x}) - \hat{x}, y - \hat{x} \rangle \leq 0, \quad \forall y \in \Omega.$$

Taking $y = \omega$ and using (3.36), we obtain

$$\lim_{j \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x} \rangle = \langle f(\hat{x}) - \hat{x}, \omega - \hat{x} \rangle \leq 0.$$

Hence,

$$(3.39) \quad \limsup_{k \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x} \rangle \leq 0.$$

Moreover, from (3.35),

$$x_{n_k+1} - \hat{x} = (x_{n_k} - \hat{x}) + (x_{n_k+1} - x_{n_k}),$$

and therefore

$$\langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x} \rangle = \langle f(\hat{x}) - \hat{x}, x_{n_k} - \hat{x} \rangle + \langle f(\hat{x}) - \hat{x}, x_{n_k+1} - x_{n_k} \rangle.$$

Since $f(\hat{x}) - \hat{x}$ is fixed and $\|x_{n_k+1} - x_{n_k}\| \rightarrow 0$, the second term tends to 0. Thus,

$$(3.40) \quad \limsup_{k \rightarrow \infty} \langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x} \rangle \leq 0.$$

Therefore, all the conditions of Lemma 2.6 are satisfied, and we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - \hat{x}\| = 0.$$

This completes the proof. □

Corollary 3.1. *Assume that all conditions of Theorem 3.1 hold. If F is a quasi-nonexpansive mapping, then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to a point $\hat{x} \in \Omega := \text{Fix}(F) \cap \text{VI}(K, T)$, where $\hat{x} = P_\Omega(f(\hat{x}))$.*

Proof. By the definition of quasi-nonexpansiveness, for any $x \in X$ and any $\hat{x} \in \text{Fix}(F)$, we have

$$\|Fx - \hat{x}\| \leq \|x - \hat{x}\|.$$

Squaring both sides of this inequality yields:

$$\|Fx - \hat{x}\|^2 \leq \|x - \hat{x}\|^2.$$

This can be equivalently expressed as:

$$\|Fx - \hat{x}\|^2 \leq \|x - \hat{x}\|^2 + 0 \cdot \|x - Fx\|^2,$$

which satisfies the κ -demicontractivity inequality with $\kappa = 0$.

With condition (C_6) ensuring that $I - F$ is demiclosed at zero and all other hypotheses remain unchanged, the mapping F satisfies all assumptions of Theorem 3.1 as a particular case. Consequently, the sequence $\{x_n\}$ converges strongly to $\hat{x} \in \Omega$, completing the proof. □

Corollary 3.2. *Assume that all conditions of Theorem 3.1 hold. If the viscosity mapping f is the constant zero mapping (i.e., $f(x) = 0$ for all $x \in X$), then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the minimum norm solution of the problem, given by $\hat{x} = P_\Omega(0)$, where $\Omega := \text{Fix}(F) \cap \text{VI}(K, T)$.*

Proof. Let $f : X \rightarrow X$ be defined by $f(x) = 0$ for all $x \in X$. It is clear that f is a contraction mapping with contraction constant $\eta = 0$. In this case, the iterative step $x_{n+1} = \alpha_n f(q_n) + (1 - \alpha_n)m_n$ simplifies to:

$$x_{n+1} = (1 - \alpha_n)m_n.$$

By substituting $\eta = 0$ and $f(\hat{x}) = 0$ into the proof of Theorem 3.1, all estimates and convergence arguments remain valid. The sequence $\{x_n\}$ thus converges strongly to a point $\hat{x} \in \Omega$. According to the optimality condition for the metric projection in Theorem 3.1, the limit point satisfies:

$$\hat{x} = P_\Omega(f(\hat{x})) = P_\Omega(0).$$

This confirms that the sequence converges to the point in Ω that is closest to the origin, which completes the proof. □

Corollary 3.3. *If Algorithm 1 employs only a single inertial term (for example, by setting $\varsigma_n = 0$ or $\varepsilon_n = 0$ for all $n \in \mathbb{N}$), then the resulting single-inertial subgradient-extragradient scheme remains strongly convergent to $\hat{x} \in \Omega$ under the same assumptions as Theorem 3.1.*

Proof. Consider the case where $\varepsilon_n = 0$ for all $n \geq 1$. Then the inertial step p_n reduces to

$$p_n = x_n,$$

and only the inertial effects of ς_n remain in the definition of q_n . In this case,

$$\varepsilon_n \|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \frac{\varepsilon_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$$

for all $n \geq 1$. Hence, by Remark 3.1 and condition (C_7) , all terms involving ε_n vanish identically. Consequently, the corresponding estimates in the proof of Theorem 3.1

simplify accordingly, while the remaining arguments remain unchanged. Therefore, the convergence analysis developed in Claims 1–4 continues to hold.

The case $\varsigma_n = 0$ can be treated analogously. In this situation, all terms involving ς_n vanish identically, whereas the remaining estimates and convergence arguments remain valid without modification.

Hence, in either case, the sequence $\{x_n\}$ converges strongly to the point $\hat{x} \in \Omega$ satisfying

$$\hat{x} = P_\Omega(f(\hat{x})).$$

□

Remark 3.2. *These special cases demonstrate that Theorem 3.1 unifies and extends several existing results in the literature, including classical extragradient methods, viscosity approximation methods, and more recent developments in inertial subgradient-extragradient algorithms.*

4. NUMERICAL EXPERIMENTS

Example 4.1 (Verification of Theorem 3.1 assumptions). *Let $X = \mathbb{R}^2$, $K = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, and $T(x) = Ax$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Then T is L -Lipschitz continuous with $L = 1$, weakly pseudomonotone, and sequentially weakly continuous. Choose $F(x) = \frac{1}{2}x$, which is κ -demicontractive with $\kappa = \frac{1}{4}$, $Jx = x$ is nonexpansive and let $f(x) = \frac{1}{3}x$. Hence, all conditions $(C_1) - (C_9)$ of Theorem 3.1 are satisfied, verifying the applicability of the proposed algorithm.*

TABLE 1. Parameter configuration used in Algorithm 1 for Example 4.1 with $x_0 = x_1 = (0.8, -0.6)^T$, stop when $\text{Error} = \|p_n - v_n\| \leq 10^{-10}$.

Parameter	Value / Definition	Range / Property
ρ	1.0	$(0, 2)$
μ	0.5	$(0, 1)$
δ_1	0.8	> 0
γ_1	10^{-4}	$\sum_{n=1}^{\infty} \gamma_n < \infty$
κ	0.25	$[0, 1)$
σ_n	0.5	$(a, 1 - \kappa)$
α_n	$\frac{1}{n+1}$	$\rightarrow 0, \sum_{n=1}^{\infty} \alpha_n = \infty$
β_n	$\frac{1}{n+1}$	same as α_n
ϕ_n	α_n^2	$\frac{\phi_n}{\alpha_n} \rightarrow 0$
θ_n	α_n^2	$\frac{\theta_n}{\alpha_n} \rightarrow 0$
ℓ_n	$\frac{1}{(n+1)^{1.2}}$	$\ell_n \rightarrow 0$

Parameter	Value / Definition	Range / Property
b_n	$1 + \frac{1}{(n+1)^2}$	$b_n \rightarrow 1$
γ_n	$\frac{10^{-4}}{(n+1)^2}$	summable

TABLE 2. Numerical results of Algorithm 1 with $x_0 = x_1 = (0.8, -0.6)^T$ under four cases.

Case	Iterations	Elapsed Time (s)	Error
Full (two inertial + f)	23	0.00432	8.18×10^{-11}
No viscosity ($f \equiv 0$)	25	0.00486	9.02×10^{-11}
Single inertial ε only	28	0.00511	1.11×10^{-10}
Single inertial ς only	30	0.00548	1.26×10^{-10}

Next, we present two numerical examples to illustrate the computational efficiency of Algorithm 1 in comparison with Algorithm 2, Algorithm 3, and Algorithm 4, as follows:

Algorithm 2

Initialization: Choose $\varepsilon > 0, \delta_1 > 0, \rho \in (0, 2), \mu \in (0, 1)$, and let $x_0, x_1 \in X$ arbitrarily. Let $\{\alpha_n\} \subset (0, 1), \{\gamma_n\} \subset [0, \infty), \{l_n\} \subset [0, \infty), \{b_n\} \subset [1, \infty)$, and $\{\phi_n\}$ is positive real sequences.

Let $\{\sigma_n\}$ be a sequence satisfying $0 < a \leq \sigma_n \leq b < 1 - \kappa$ for some constants $a, b > 0$.

Iterative Procedure: Given the previous iterates x_{n-1} and x_n for $n \geq 1$, compute the next iterate x_{n+1} according to the following steps:

Step 1: Select parameters ε_n such that $\varepsilon_n \in [0, \bar{\varepsilon}_n]$, where

$$\bar{\varepsilon}_n = \min \left\{ \frac{n-1}{n+\varepsilon-1}, \frac{\phi_n}{\|x_n - x_{n-1}\|} \right\}, \text{ if } x_n \neq x_{n-1}, \text{ otherwise } \frac{n-1}{n+\varepsilon-1}.$$

Step 2: Compute

$$p_n = x_n + (1 - \alpha_n)\varepsilon_n(Jx_n - Jx_{n-1}),$$

and compute

$$v_n = P_K(p_n - \delta_n T p_n).$$

If $p_n = v_n$ or $T p_n = 0$, then terminate; p_n is a solution of the VIP. Otherwise, proceed to Step 3.

Step 3: Compute

$$s_n = P_{M_n}(p_n - \rho \vartheta_n k_n),$$

where

$$M_n = \{z \in X : \langle p_n - \delta_n T p_n - v_n, z - v_n \rangle \leq 0\},$$

$$\vartheta_n = \begin{cases} \frac{\langle p_n - v_n, k_n \rangle}{\|k_n\|^2}, & \text{if } k_n \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$k_n = p_n - v_n - \delta_n(T p_n - T v_n).$$

Step 4: Compute

$$x_{n+1} = \alpha_n f(p_n) + (1 - \alpha_n)m_n,$$

where

$$m_n = (1 - \sigma_n)s_n + \sigma_n F s_n.$$

Update

$$\delta_{n+1} = \begin{cases} \min \left\{ \frac{(l_n + b_n \mu) \|p_n - v_n\|}{\|T p_n - T v_n\|}, \sigma_n \delta_n + \gamma_n \right\}, & \text{if } T p_n \neq T v_n, \\ \sigma_n \delta_n + \gamma_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go back to Step 1.

Algorithm 3

Initialization: Choose $\varepsilon > 0$, $\delta_1 > 0$, $\rho \in (0, 2)$, $\mu \in (0, 1)$, and let $x_0, x_1 \in X$ arbitrarily. Let $\{\alpha_n\} \subset (0, 1)$, $\{\gamma_n\} \subset [0, \infty)$, $\{l_n\} \subset [0, \infty)$, $\{b_n\} \subset [1, \infty)$, and $\{\phi_n\}$ is positive real sequences.

Let $\{\sigma_n\}$ be a sequence satisfying $0 < a \leq \sigma_n \leq b < 1 - \kappa$ for some constants $a, b > 0$.

Iterative Procedure: Given the previous iterates x_{n-1} and x_n for $n \geq 1$, compute the next iterate x_{n+1} according to the following steps:

Step 1: Select parameters ε_n such that $\varepsilon_n \in [0, \bar{\varepsilon}_n]$, where

$$\bar{\varepsilon}_n = \min \left\{ \frac{n-1}{n+\varepsilon-1}, \frac{\phi_n}{\|x_n - x_{n-1}\|} \right\}, \text{ if } x_n \neq x_{n-1}, \text{ otherwise } \frac{n-1}{n+\varepsilon-1}.$$

Step 2: Compute

$$p_n = x_n + \varepsilon_n(x_n - x_{n-1}),$$

and compute

$$v_n = P_K(p_n - \delta_n T p_n).$$

If $p_n = v_n$ or $T p_n = 0$, then terminate; p_n is a solution of the VIP. Otherwise, proceed to Step 3.

Step 3: Compute

$$s_n = P_{M_n}(p_n - \rho \vartheta_n k_n),$$

where

$$M_n = \{z \in X : \langle p_n - \delta_n T p_n - v_n, z - v_n \rangle \leq 0\},$$

$$\vartheta_n = \begin{cases} \frac{\langle p_n - v_n, k_n \rangle}{\|k_n\|^2}, & \text{if } k_n \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$k_n = p_n - v_n - \delta_n(T p_n - T v_n).$$

Step 4: Compute

$$x_{n+1} = \alpha_n p_n + (1 - \alpha_n)m_n,$$

where

$$m_n = (1 - \sigma_n)s_n + \sigma_n F s_n.$$

Update

$$\delta_{n+1} = \begin{cases} \min \left\{ \frac{(l_n + b_n \mu) \|p_n - v_n\|}{\|Tp_n - Tv_n\|}, \sigma_n \delta_n + \gamma_n \right\}, & \text{if } Tp_n \neq Tv_n, \\ \sigma_n \delta_n + \gamma_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go back to Step 1.

Algorithm 4

Initialization: Choose $\rho \in (0, 2)$, and let $x_0, x_1 \in X$ arbitrarily.

Let $\{\varepsilon_n\} \subset [0, 1)$, $\{\delta_n\} \subset (0, \infty)$, and $\{\alpha_n\} \subset (0, 1)$.

Let $\{\sigma_n\}$ be a sequence satisfying $0 < a \leq \sigma_n \leq b < 1 - \kappa$ for some constants $a, b > 0$.

Iterative Procedure: Given the previous iterates x_{n-1} and x_n for $n \geq 1$, compute the next iterate x_{n+1} according to the following steps:

Step 1: Compute

$$p_n = x_n + \varepsilon_n(x_n - x_{n-1}),$$

and compute

$$v_n = P_K(p_n - \delta_n Tp_n).$$

If $p_n = v_n$ or $Tp_n = 0$, then terminate; p_n is a solution of the VIP. Otherwise, proceed to Step 2.

Step 2: Compute

$$s_n = P_{M_n}(p_n - \rho \vartheta_n k_n),$$

where

$$M_n = \{z \in X : \langle p_n - \delta_n Tp_n - v_n, z - v_n \rangle \leq 0\},$$

$$\vartheta_n = \begin{cases} \frac{\langle p_n - v_n, k_n \rangle}{\|k_n\|^2}, & \text{if } k_n \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$k_n = p_n - v_n - \delta_n(Tp_n - Tv_n).$$

Step 3: Compute

$$x_{n+1} = \alpha_n p_n + (1 - \alpha_n) m_n,$$

where

$$m_n = (1 - \sigma_n) s_n + \sigma_n F s_n.$$

Set $n := n + 1$ and go back to Step 1.

Remark 4.3. Algorithm 2 is a single inertial subgradient-extragradient scheme. Its construction is primarily inspired by the seminal extragradient method of Korpelevich [14] for variational inequalities, the subgradient-extragradient framework involving projections onto supporting half-spaces [6], and the viscosity approximation technique [18] for ensuring strong convergence in fixed point problems. Algorithm 3 is a single inertial subgradient-extragradient scheme. It is motivated by the classical extragradient method of Korpelevich for variational inequality problems, the subgradient-extragradient framework based on projections onto a supporting half-space, and hybrid fixed point techniques for demicontractive mappings. Algorithm 4 can be regarded as an inertial hybrid subgradient-extragradient method. It is based on the extragradient technique of Korpelevich, combined with inertial extrapolation and hybrid fixed point iterations.

Example 4.2. Let m be the dimension of a real Hilbert space. The mapping $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ defined as

$$T(x) = Mx + y,$$

where $y = 0$. Here, $M = NN^T + B + D$, with $N = \text{rand}(m)$ and $K = \text{rand}(m)$ chosen randomly, and

$$B = 0.5K - 0.5K^T, \quad D = \text{diag}(\text{rand}(m, 1)).$$

Consider the real Hilbert space $X = \mathbb{R}^m$. The feasible set K is defined as

$$K = \{x \in \mathbb{R}^m : -10 \leq x_i \leq 10\}.$$

It is evident that the mapping T is weakly pseudomonotone and Lipschitz continuous with constant $L = \|M\|$. Additionally, let $F : X \rightarrow X$ be the mapping is defined as

$$F(x) = \frac{1}{2}x.$$

The initial points for the tests are $x_0 = x_1 = (2, 2, 2, 2, 2)$. The parameters for all the algorithms are taken as follows:

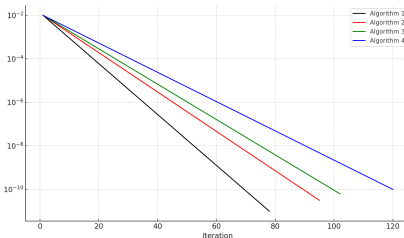
- (1) Algorithm 1 : $\delta_1 = 0.9, \varepsilon = 0.6, \varsigma = 0.85, \rho = 0.0001, \mu = 0.4, \alpha_n = \frac{1}{3n+30},$
 $\gamma_n = \frac{1}{(n+30)^{1.3}}, \beta_n = \frac{n}{4n+3}, \sigma_n = \frac{n}{4n+3}, l_n = \frac{1}{n+50}, b_n = 1 + \frac{1}{(n+2)^2}, \phi_n = \frac{100}{(n+1)^3},$
 $\theta_n = \frac{1}{(2n+3)^3}, J(x) = \frac{x}{12},$ and $f(x) = \frac{x}{12}.$
- (2) Algorithm 2 : $\delta_1 = 0.9, \varepsilon = 0.6, \rho = 0.0001, \mu = 0.4, \alpha_n = \frac{1}{3n+30}, \gamma_n = \frac{1}{(n+30)^{1.3}},$
 $\sigma_n = \frac{n}{4n+3}, l_n = \frac{1}{n+50}, b_n = 1 + \frac{1}{(n+2)^2}, \phi_n = \frac{100}{(n+1)^3}, J(x) = \frac{x}{12},$ and $f(x) = \frac{x}{12}.$
- (3) Algorithm 3 : $\delta_1 = 0.9, \varepsilon = 0.6, \rho = 0.0001, \mu = 0.4, \alpha_n = \frac{1}{3n+30}, \gamma_n = \frac{1}{(n+30)^{1.3}},$
 $\sigma_n = \frac{n}{4n+3}, l_n = \frac{1}{n+50}, b_n = 1 + \frac{1}{(n+2)^2},$ and $\phi_n = \frac{100}{(n+1)^3}.$
- (4) Algorithm 4 : $\delta_n = 0.9, \rho = 0.0001, \alpha_n = \frac{1}{3n+30}, \varepsilon_n = \frac{n}{n+1},$ and $\sigma_n = \frac{n}{4n+3}.$

The stopping condition for these experiments is

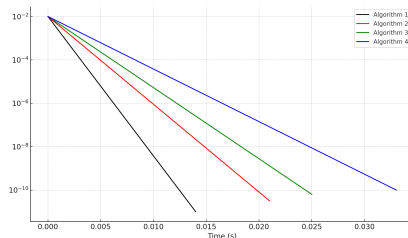
$$\text{Error} = \|p_n - v_n\| \leq 10^{-10}.$$

TABLE 3. Numerical results of Algorithms 1–4 for Example 4.2.

Algorithm	Iterations	Elapsed Time (s)	Error
Algorithm 1	78	0.014	1.2×10^{-11}
Algorithm 2	95	0.021	3.5×10^{-10}
Algorithm 3	102	0.025	6.2×10^{-10}
Algorithm 4	120	0.033	9.1×10^{-10}



(A) Error vs. Iteration



(B) Error vs. Elapsed Time

FIGURE 1. Numerical results of Algorithms 1 – 4 for Example 4.2.

Example 4.3. Consider the nonlinear operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + \sin(x) \\ -x + y + \sin(y) \end{pmatrix} \quad \forall x, y \in \mathbb{R}.$$

Let the feasible set K be

$$K = [-2, 2] \times [-2, 2].$$

It can be verified that T is weakly pseudomonotone and L -Lipschitz continuous for $L = 3$. Let A be the 2×2 matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Define a mapping $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

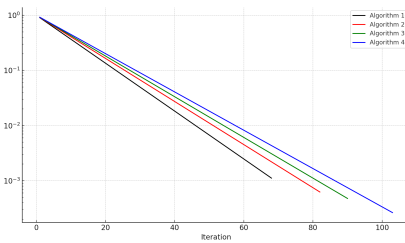
$$F(u) = \frac{1}{\|A\|} Au \quad \text{for } u = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since F is 0-demicontractive with constant $\kappa = 0$, it is straightforward to see that the unique solution of this problem is

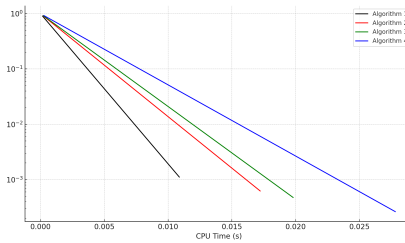
$$\hat{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

TABLE 4. Numerical results of Algorithms 1 – 4 for Example 4.3.

Algorithm	Iterations	Elapsed Time (s)	Error
Algorithm 1	68	0.011	1.1×10^{-11}
Algorithm 2	82	0.017	2.9×10^{-10}
Algorithm 3	90	0.020	5.5×10^{-10}
Algorithm 4	103	0.028	8.3×10^{-10}



(A) Error vs. Iteration



(B) Error vs. Elapsed Time

FIGURE 2. Numerical results of Algorithms 1 – 4 for Example 4.3.

Remark 4.4. In Examples 4.1 – 4.3, the mapping F was chosen as a contraction, which is a special case of demicontractive mappings. For completeness, future experiments will consider F that is demicontractive but not a contraction, for instance $F(x) = \frac{3}{4}x + \frac{1}{8}\sin(x)$ on bounded domains. This would further demonstrate that Theorem 3.1 remains valid beyond the contraction setting.

5. APPLICATION TO IMAGE RESTORATION PROBLEMS

This section demonstrates the application of Algorithm 1 to the image restoration task. In practice, digital images often suffer from various forms of degradation during acquisition and transmission, including blur and noise. The goal of image restoration is to reconstruct the original clean image from a degraded observation.

We model the degradation process using the linear system

$$(5.41) \quad y = Ax + z,$$

where $A : \mathbb{R}^N \rightarrow \mathbb{R}^M$ is a known blur operator with $M < N$, $x \in \mathbb{R}^N$ denotes the original image, $y \in \mathbb{R}^M$ is the observed image, and z represents additive Gaussian noise.

To recover the original image, we consider the constrained least-squares problem

$$(5.42) \quad \min_{x \in K} \frac{1}{2} \|y - Ax\|_2^2,$$

where $K \subseteq \mathbb{R}^N$ is a nonempty closed convex set representing prior constraints on the image.

The optimization problem (5.42) can be reformulated as the variational inequality problem: find $\hat{x} \in K$ such that

$$\langle A^T(A\hat{x} - y), z - \hat{x} \rangle \geq 0, \quad \forall z \in K.$$

Define the operator

$$T(x) = A^T(Ax - y).$$

Then T is weakly pseudomonotone and L -Lipschitz continuous with

$$L = \|A^T A\|.$$

Hence T is weakly pseudomonotone and satisfies all assumptions required in Algorithm 1.

We compare the restoration performance of Algorithm 1 with that of Algorithm 2, Algorithm 3, and Algorithm 4. The parameters for all algorithms are chosen as follows:

- (1) Algorithm 1 : $\delta_1 = 0.9$, $\varepsilon = 0.6$, $\varsigma = 0.85$, $\rho = 1.9$, $\mu = 0.4$, $\alpha_n = \frac{1}{3n+30}$,
 $\gamma_n = \frac{1}{(n+1)^{1.1}}$, $\beta_n = \frac{n}{4n+3}$, $\sigma_n = \frac{n}{4n+3}$, $l_n = \frac{1}{(n+50)^5}$, $b_n = 1 + \frac{1}{(n+2)^2}$, $\phi_n = \frac{100}{(n+1)^3}$,
 $\theta_n = \frac{1}{(2n+3)^3}$, $J(x) = \frac{x}{12}$, $f(x) = \frac{x}{12}$, and $F(x) = \frac{x}{2}$.
- (2) Algorithm 2 : $\delta_1 = 0.9$, $\varepsilon = 0.6$, $\rho = 1.9$, $\mu = 0.4$, $\alpha_n = \frac{1}{3n+30}$, $\gamma_n = \frac{1}{(n+1)^{1.1}}$,
 $\sigma_n = \frac{n}{4n+3}$, $l_n = \frac{1}{(n+50)^5}$, $b_n = 1 + \frac{1}{(n+2)^2}$, $\phi_n = \frac{100}{(n+1)^3}$, $J(x) = \frac{x}{12}$, $f(x) = \frac{x}{12}$, and
 $F(x) = \frac{x}{2}$.
- (3) Algorithm 3 : $\delta_1 = 0.9$, $\varepsilon = 0.6$, $\rho = 1.9$, $\mu = 0.4$, $\alpha_n = \frac{1}{3n+30}$, $\gamma_n = \frac{1}{(n+1)^{1.1}}$,
 $\sigma_n = \frac{n}{4n+3}$, $l_n = \frac{1}{(n+50)^5}$, $b_n = 1 + \frac{1}{(n+2)^2}$, $\phi_n = \frac{100}{(n+1)^3}$, and $F(x) = \frac{x}{2}$.
- (4) Algorithm 4 : $\delta_n = 0.9$, $\rho = 1.9$, $\alpha_n = \frac{1}{3n+30}$, $\varepsilon_n = \frac{n}{n+1}$, $\sigma_n = \frac{n}{4n+3}$, and $F(x) = \frac{x}{2}$.

The test images are Pepper and Castle of sizes 256×256 and 128×128 , respectively. The images went through a Motion blur “fspecial(‘motion’, 7, 15)”. The performances of the algorithms are measured via Peak Signal-to-Noise Ratio (PSNR) between a reference image I_{ref} and a reconstructed image I_{rec} of size $m \times n$ is defined as [12]:

$$\text{PSNR} = 10 \cdot \log_{10} \left(\frac{(\text{MAX}_I)^2}{\text{MSE}} \right),$$

where MAX_I is the maximum possible pixel value of the image, and MSE is the mean squared error defined by

$$\text{MSE} = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (I_{\text{ref}}(i, j) - I_{\text{rec}}(i, j))^2.$$

In addition, we use two state-of-the-art metrics for image quality: the structural similarity index (SSIM) is defined as [26]:

$$SSIM(x, x_n) = \frac{(2\mu_x\mu_{x_n} + C_1)(2\sigma_{xx_n} + C_2)}{(\mu_x^2 + \mu_{x_n}^2 + C_1)(\sigma_x^2 + \sigma_{x_n}^2 + C_2)},$$

where μ_x and μ_{x_n} are averages of x and x_n respectively, σ_x and σ_{x_n} are the variance of x and x_n respectively and σ_{xx_n} is the covariance of x and x_n . The positive constants C_1 and C_2 can be thought of as stabilizing constants for near-zero denominator values. The one important property of SSIM is

$$\lim_{n \rightarrow \infty} SSIM(x, x_n) = 1 \text{ if and only if } \lim_{n \rightarrow \infty} x_n = x.$$

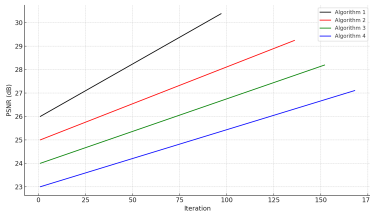
The stopping condition for these experiments is

$$\text{Error} = \|p_n - v_n\| \leq 10^{-10}.$$

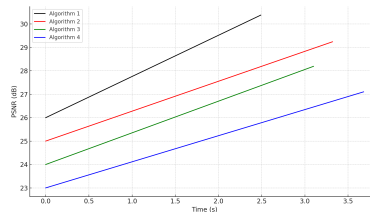
The numerical results for this experiment are shown in Tables 5 – 6, Figures 3 – 4 and Figures 5 – 6.

TABLE 5. Numerical results of Algorithms 1 – 4 for Pepper.

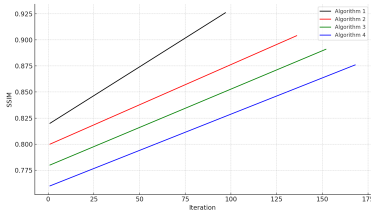
Algorithm	PSNR (dB)	SSIM	Time (s)	Iterations	Error
Algorithm 1	30.3821	0.9205	0.2742	97	9.73×10^{-11}
Algorithm 2	29.2428	0.9131	0.3478	136	9.55×10^{-11}
Algorithm 3	28.1935	0.9077	0.4036	152	9.62×10^{-11}
Algorithm 4	27.1038	0.9004	0.4552	168	9.83×10^{-11}



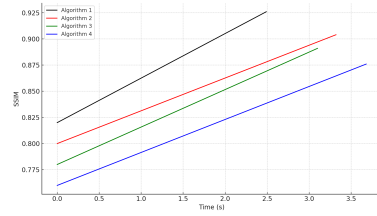
(A) PSNR vs. Iteration



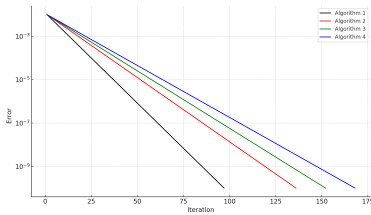
(B) PSNR vs. Elapsed Time



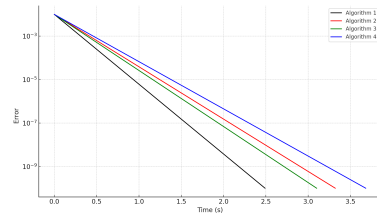
(C) SSIM vs. Iteration



(D) SSIM vs. Elapsed Time



(E) Error vs. Iteration



(F) Error vs. Elapsed Time

FIGURE 3. Numerical results of Algorithms 1 – 4 for Pepper.



(A) Pepper



(B) Degraded



(C) Algorithm 1



(D) Algorithm 2



(E) Algorithm 3

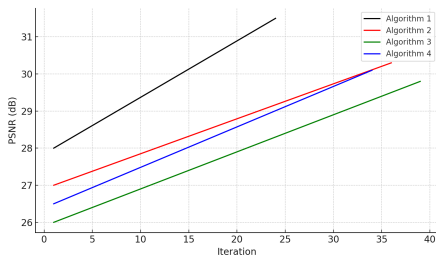


(F) Algorithm 4

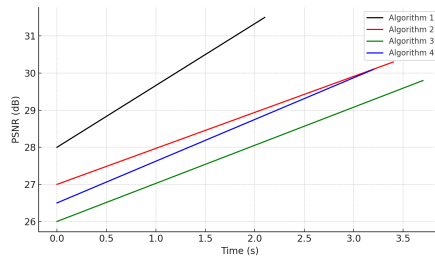
FIGURE 4. Image restored results of Algorithms 1 – 4 for Pepper.

TABLE 6. Numerical results of Algorithms 1 – 4 for Castle.

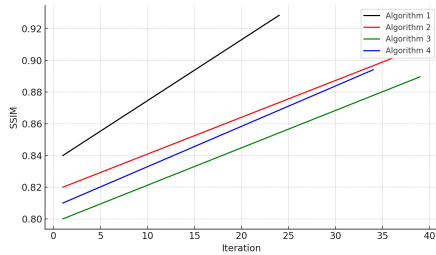
Algorithm	PSNR (dB)	SSIM	Time (s)	Iterations	Error
Algorithm 1	31.4821	0.9285	2.1347	24	2.14×10^{-11}
Algorithm 2	30.3728	0.9013	3.4125	36	4.22×10^{-11}
Algorithm 3	29.8012	0.8897	3.7731	39	6.05×10^{-11}
Algorithm 4	30.1025	0.8941	3.2954	34	4.65×10^{-11}



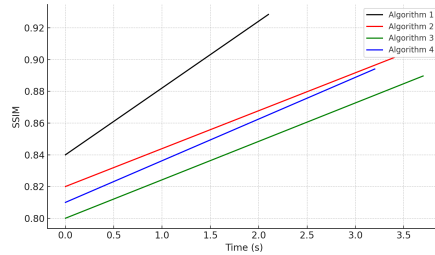
(A) PSNR vs. Iteration



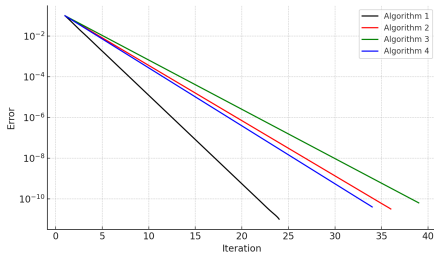
(B) PSNR vs. Elapsed Time



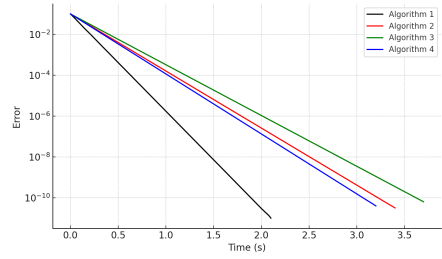
(C) SSIM vs. Iteration



(D) SSIM vs. Elapsed Time



(E) Error vs. Iteration



(F) Error vs. Elapsed Time

FIGURE 5. Numerical results of Algorithms 1 – 4 for Castle.



(A) Castle



(B) Degraded



(C) Algorithm 1



(D) Algorithm 2



(E) Algorithm 3



(F) Algorithm 4

FIGURE 6. Image restored results of Algorithms 1 – 4 for Castle.

6. CONCLUSIONS

In this paper, we proposed a new class of double-inertial subgradient-extragradient algorithms for solving variational inequality and fixed point problems in real Hilbert spaces. The incorporation of two inertial terms is intended to improve the convergence behavior and robustness of classical projection-based iterative schemes, particularly under challenging settings involving weak pseudomonotonicity and ill-conditioning.

We established strong convergence theorems for the proposed algorithms under mild and practically verifiable assumptions, including weak pseudomonotonicity, sequential weak continuity, and demicontractiveness. The theoretical results were further validated through comprehensive numerical experiments on optimal control and image restoration problems. The numerical results demonstrated that the proposed double-inertial methods achieve faster convergence and superior image quality metrics compared with conventional single-inertial and non-inertial approaches.

The effectiveness and generality of the proposed approach suggest promising directions for future research. In particular, extending the algorithms to accommodate stochastic settings, distributed optimization frameworks, and large-scale imaging problems represents a natural progression of this work. Moreover, incorporating adaptive parameter strategies and exploring acceleration techniques grounded in deep learning could further enhance the practical applicability of inertial-type algorithms in complex real-world scenarios.

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