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# Parametric linear systems in extremal algebras

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ABSTRACT. We study systems of linear equations $A(p) \otimes x = b(p)$ in extremal algebras (the standard pair of operations plus and times is substituted by the pair of operations maximum denoted by $\oplus$ and either plus or minimum denoted by $\otimes$), where the entries of the matrix and of the right-hand side vector are linear dependent on parameters.

A parametric system of linear equations $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is the set of all parametric systems of the form $A(p) \otimes x = b(p)$ for some $p \in \boldsymbol{p}$ and $\boldsymbol{p}$ is an interval vector. A parametric system of the form $A(p) \otimes x = b(p)$ is called a parametric subsystem of parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ if $p \in \boldsymbol{p}$. If we ask for the solvability of at least one of subsystems we say about the possible solvability and the universal solvability which requires solvability of all subsystems. This article deals with the universal and possible solvability of parametric linear equations. In addition, four other versions derived from them are studied, namely tolerable EA-solvability, tolerable AE-solvability, controllable AE-solvability, and weak EE-solvability. For each concept of solvability of parametric linear equations, we present equivalent conditions, some of which are polynomially checked. Presented numerical examples illustrate motivation models and properties a tolerable AE-solvability.

## 1. BACKGROUND OF THE PROBLEM

The research of extremal algebras (the standard pair of operations plus and times is substituted by the pair of operations maximum denoted by $\oplus$ and either plus (called max-plus algebra) or minimum (called max-min algebra) denoted by $\otimes$) can be motivated by various versions of discrete event dynamic systems which have been described by max-plus (max-min) linear systems, later, by linear independences, regularity and bideterminants [1], [3]–[5], [8], [12], [14].

The following model is a simple instance of the application of max-plus parametric linear equations.

Consider the max-plus version of parallel factory system $S$ consists of $k$ simple *multi-machine interactive production processes* $S_1, \ldots, S_k$ presented in [5], [8]: In each of these processes we have $n$ machines $M_1, \ldots, M_n$ producing the products $Q_1, \ldots, Q_m$. Each machine of $S_t$, $t = 1, \ldots, k$ is contributing to the completion of each product and working for all product simultaneously. In the algebraic model of their interactive work in the system $S_t$, entry $x_j$ represents the starting time of the $j$th machine and the entry $a_{ij}^{(t)}$ of a matrix $A^{(t)}$ encodes the duration of the work of the $j$th machine needed to complete the partial product $Q_i$. If this relation is not required for some $i, j$, put $a_{ij}^{(t)} := -\infty$. Then all partial products for $Q_i$ will be ready at time

$$\max(x_1 + a_{i1}^{(t)}, \ldots, x_n + a_{in}^{(t)}) \text{ for } i = 1, \ldots, m. \tag{1.1}$$

In addition, suppose that $b_1^{(t)}, \ldots, b_m^{(t)}$ are required completion times. Then the starting times in all systems $S_t$ have to satisfy the max-plus linear system:

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(1.2) $$\max(x_1 + a_{i1}^{(t)}, \dots, x_n + a_{in}^{(t)}) = b_i^{(t)} \text{ for } i = 1, \dots, m \text{ and } t = 1, \dots, k.$$

Observe that (1.2) can be rewritten as max-plus matrix-vector form $A^{(t)} \otimes x = b^{(t)}$ for $t = 1, \dots, k$. The simple version of this model for $n = 3,\ m = 4$ is displays in Fig. 1.

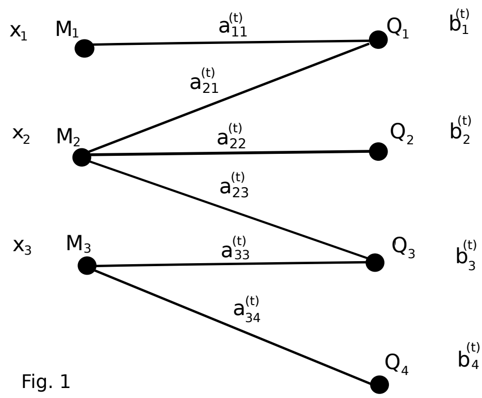

Fig. 1

Suppose that an error $e_t$ occurs in the system $S_t$, causing the entire system to stop for the necessary repair time $p_t$. Thus, all inputs to the system will change by the same value $p_t$, i.e. all entries of $S_t$ will be changed by the same measure and we will consider instead of $S_t$ the modified process, say $S'_t$, with new entries ${a'}_{ij}^{(t)} := a_{ij}^{(t)} + p_t$, $i = 1, \dots, m$, $j = 1, \dots, n$ and ${b'}_i^{(t)} := b_i^{(t)} + p_t$ for $i = 1, \dots, m$ and which can be written as

$$A'^{(t)} \otimes x = b'^{(t)} \Leftrightarrow p_t \otimes A^{(t)} \otimes x = p_t \otimes b^{(t)}.$$

The target is to find multi-machine interactive production process $S$ consists of new entries $a_{ij} := \bigoplus_{t=1}^{k} (a_{ij}^{(t)} + p_t)$, $i = 1, \dots, m$, $j = 1, \dots, n$ and $b_i := \bigoplus_{t=1}^{k} (b_i^{(t)} + p_t)$ for $i = 1, \dots, m$ with potential errors $e_1, \dots, e_k$ (see Fig. 2) and which guarantees that all partial products $Q_1, \dots, Q_m$ will be ready at time with respect to delay of completion times, i.e., we look for parameters $p_1, \dots, p_k$ and starting times $x_1, \dots, x_n$ such that

$$(\bigoplus_{t=1}^{k} p_t \otimes A^{(t)}) \otimes x = \bigoplus_{t=1}^{k} p_t \otimes b^{(t)}.$$

Using max-plus algebra this system can be written in a compact form as a system of parametric linear equations $A \otimes x = b$, where $A = \bigoplus_{t=1}^{k} p_t \otimes A^{(t)}$ and $b = \bigoplus_{t=1}^{k} p_t \otimes b^{(t)}$.

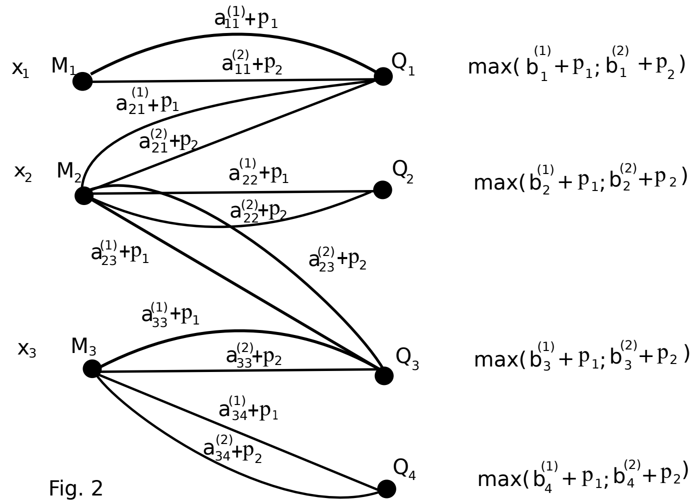

Fig. 2

Suppose now that $t = 2$ and matrices $A_1,\ A_2$, vectors $b_1,\ b_2$ and the interval vector $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2)$ have the following forms:

$$A_1 = \begin{pmatrix} 1 & \varepsilon & \varepsilon & \varepsilon \\ 2 & 1 & 3 & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 4 & \varepsilon \end{pmatrix},\ A_2 = \begin{pmatrix} 2 & \varepsilon & \varepsilon & \varepsilon \\ 4 & 4 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 3 & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix},\ b_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix},\ b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ \boldsymbol{p} = \begin{pmatrix} [0,2] \\ [1,5] \end{pmatrix}.$$

Then the parametric linear system can be written as follows

$$\left[ p_1 \otimes \begin{pmatrix} 1 & \varepsilon & \varepsilon & \varepsilon \\ 2 & 1 & 3 & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & 4 & \varepsilon \end{pmatrix} \oplus p_2 \otimes \begin{pmatrix} 2 & \varepsilon & \varepsilon & \varepsilon \\ 4 & 4 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 3 & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix} \right] \otimes x = p_1 \otimes \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \oplus p_2 \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and the task is to find parameters $p_1 \in \boldsymbol{p}_1$, $p_2 \in \boldsymbol{p}_2$ and vector $x$. Observe that for $p_1 = 1$, $p_2 = 2$ the parametric system has the form

$$\begin{pmatrix} 4 & \varepsilon & \varepsilon & \varepsilon \\ 6 & 6 & 4 & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon \end{pmatrix} \otimes x = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 2 \end{pmatrix}$$

and one of its solutions is vector $x = (-2, -2, -3, 0)^T$.

Systems of linear equations often describe the behavior of discrete-event systems in which the components move from event to event, rather than varying continuously over time. Discrete event dynamic systems and related algebraic structures have been studied using max-plus and max-min matrix operations. Systems of linear equations in max-plus and max-min algebras of the form $A \otimes x = b$, where $A$ is a matrix, $b$ and $x$ are vectors of compatible dimensions, are used in several branches of applied mathematics and practice. In the last decade, interval systems of the form $\boldsymbol{A} \otimes x = \boldsymbol{b}$, where entries of $\boldsymbol{A}$, $\boldsymbol{b}$ are intervals, have been intensively studied; for details, see [23]–[30]. Unfortunately, authors, despite great efforts, have not found any articles concerning parametric systems in max-plus and/or max-min algebras. This paper generalizes the notions of solvability of a linear system $A \otimes x = b$ with parametric interval entries and also studies six of its concepts.

Particularly, in this paper the properties of two concepts of solvability of a parametric linear system, i.e., universally and possibly parametric linear systems, are presented. The interval and parametric analysis (the values of the matrix entries are not exact numbers and usually they are contained in some intervals) offers other possibilities of applying generalized results, as interval data can be easily used in computer implementations as well as in rounding. Moreover, we assume that the interval parameters can be divided into two subsets according to a forall–exists quantification of its interval items, whereby consider next four classes of solvability of parametric linear systems denoted as tolerable EA/AE-solvability, controllable AE-solvability and weak EE-solvability. We present equivalent polynomially checking conditions for some of them.

The paper is organized as follows. In Section 2 we give definitions and the known results describing a solution and the greatest solution of $A \otimes x = b$, the properties of parametric systems of linear equations are presented in Section 3. Sections 4, 5 are devoted to results on universally and possibly solvable parametric systems. Section 6 presents results on the tolerable EA/AE-solvability, controllable AE-solvability and weak EE-solvability of parametric systems.

We conclude with a brief overview of the works on max-plus algebra to which this paper is related. Various properties of interval linear systems of the form $\boldsymbol{A} \otimes x = \boldsymbol{b}$ have been studied in [23]–[30], the bideterminant and its connections with linear dependence and independence, regularity, assignment problem and eigenvalue problems have been investigated in [1], [4], [5], [6], [16]–[18], [20].

## 2. Preliminaries

An extremal algebra is a triplet $(E, \oplus, \otimes)$, where $E$, a set equipped with two binary operations $\oplus$, $\otimes$, is a commutative semigroup with respect to $\oplus$ and $\otimes$ containing neutral elements with respect to $\oplus$ and $\otimes$ and zero element with respect to $\otimes$, and in which, formulas $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ and $x \oplus y \in \{x, y\}$ hold for each $x, y, z \in E$ ([31]).

Denote the set of real numbers by $\mathbb{R}$. The symbol $\overline{\mathbb{R}}$ will stand for $\mathbb{R} \cup \{-\infty\}$. A linearly ordered set with the least element $O$ and the greatest element $I$ denote by $\mathcal{B}$.

We will deal with two extremal algebras, namely, the max-plus algebra and the max-min algebra.

Formally, max-plus algebra and max-min algebra are defined as a triplet $(\overline{\mathbb{R}}, \oplus, \otimes)$, where $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and $a \oplus b = \max(a, b)$, $a \otimes b = \min(a, b)$, respectively.

Throughout the paper we denote $-\infty$, the neutral element with respect to $\oplus$ in max-plus algebra, by $\varepsilon$. Observe that the least element $O \in \mathcal{B}$ plays the role of the neutral element with respect to $\oplus$ and it is also a zero element with respect to $\otimes$ in max-min algebra.

Note that if we are going to denote or formulate assertions for two considered extremal algebras we use a notation $\overline{\mathbb{R}}(\mathcal{B})$.

For given $k, m, n \in \mathbb{N}$, we use the notations $K = \{1, \ldots, k\}$, $M = \{1, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$. The matrix operations over $\overline{\mathbb{R}}(\mathcal{B})$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any field.

Suppose that $m \geq 1, n \geq 1$ are given natural numbers. The set of $m \times n$ matrices over $\overline{\mathbb{R}}(\mathcal{B})$ is denoted by $\overline{\mathbb{R}}^{m \times n}(\mathcal{B}^{m \times n})$, specially the set of $m \times 1$ vectors over $\overline{\mathbb{R}}(\mathcal{B})$ is denoted by $\overline{\mathbb{R}}^{m}(\mathcal{B}^{m})$.

For $A \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $B \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$ we write $A \leq B$ if $a_{ij} \leq b_{ij}$ holds true for all $i,\, j \in N$.

Suppose that we are studying system $A \otimes x = b$ with given $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ in max-plus algebra. In what follows we shall always suppose that $b_i > \varepsilon$ for all $i \in M$ in the system $A \otimes x = b$. To give reason for this assumption, we show how to oust of $\varepsilon$ on the right-hand sides. Denote by $\hat{M} = \{i \in M;\ b_i = \varepsilon\}$ and $\hat{N} = \{j \in N;\ (\exists i \in \hat{M})\ a_{ij} > \varepsilon\}$. Then any solution $x$ of $A \otimes x = b$ has $x_j = \varepsilon$ for all $j \in \hat{N}$. Hence it is possible to omit the equations with indices from $\hat{M}$ and the columns of $A$ with indices from $\hat{N}$ since the solutions of these two systems correspond to each other by putting $x_j = \varepsilon$ for $j \in \hat{N}$. Moreover, if $b \in \mathbb{R}^m$ and $A$ contains an $\varepsilon$ row, then the system $A \otimes x = b$ has no solution. If $A$ has an $\varepsilon$ column, then the corresponding coordinate in a solution $x$ can take on any value. Thus, we may suppose without loss of generality that $A$ contains no $\varepsilon$ row and no $\varepsilon$ column and $b \in \mathbb{R}^m$.

Put

$$S(A, b) = \{x \in \overline{\mathbb{R}}^n : A \otimes x = b\}. \tag{2.3}$$

For any $j \in N$ put

$$x^*_j(A, b) = \min\{b_i - a_{ij};\ i \in M,\ a_{ij} \in \mathbb{R}\}, \tag{2.4}$$

$$M_j(A, b) = \{i \in M;\ x^*_j(A, b) = b_i - a_{ij}\} \tag{2.5}$$

for max-plus case and

$$x^*_j(A, b) = \min\{b_i;\ a_{ij} > b_i;\ i \in M,\ a_{ij} \in \mathcal{B}\} \tag{2.6}$$

(where $\min \emptyset = I$) for max-min case,

**Lemma 2.1.** [5] Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then the following three statements are equivalent:

- $S(A, b) \neq \emptyset$,
- $x^*(A, b) \in S(A, b)$,
- $\bigcup_{j\in N} M_j(A, b) = M$.

**Theorem 2.1.** [31] *Suppose given* $A \in \overline{\mathbb{R}}^{m\times n}(\mathcal{B}^{m\times n})$ *and* $b \in \overline{\mathbb{R}}^m(\mathcal{B}^m)$*. Then the system* $A \otimes x = b$ *is solvable if and only if* $x^*(A, b) \in \overline{\mathbb{R}}^n(\mathcal{B}^n)$ *is its solution.*

**Theorem 2.2.** [7] *Suppose given* $B, C \in \overline{\mathbb{R}}^{m\times n}(\mathcal{B}^{m\times n})$ *and* $b, c \in \overline{\mathbb{R}}^m(\mathcal{B}^m)$*. Then the system of inequalities*

$$B \otimes x \leq b,$$
$$C \otimes x \geq c$$

*has a solution if and only if* $C \otimes x^*(B, b) \geq c$.

Notice that the solvability of $C \otimes x^*(B, b) \geq c$ can be recognized in $O(mn)$ time.

**Lemma 2.2.** [21] *Let* $A \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b, d \in \mathbb{R}^m(\mathcal{B}^m)$ *be such that* $b \leq d$*. Then* $x^*(A, b) \leq x^*(A, d)$.

**Lemma 2.3.** [21] *Let* $A, B \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b \in \mathbb{R}^m(\mathcal{B}^m)$ *be such that* $A \leq B$*. Then* $x^*(B, b) \leq x^*(A, b)$.

## 3. Properties of parametric systems of linear equations

Similarly to [9], [10], [22] define the interval matrix $\boldsymbol{A}$ with bounds $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$ and the interval vector $\boldsymbol{b}$ with bounds $\underline{b}, \overline{b} \in \mathbb{R}^m(\mathcal{B}^m)$ as follows

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n});\ \underline{A} \le A \le \overline{A} \right\}, \quad \boldsymbol{b} = [\underline{b}, \overline{b}] = \left\{ b \in \mathbb{R}^{m}(\mathcal{B}^{m});\ \underline{b} \le b \le \overline{b} \right\}.$$

Moreover, suppose that an interval vector of parameters

$$\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k), \ \ where \ \ \boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i] \subseteq \overline{\mathbb{R}}(\mathcal{B})$$

is given. A linear parametric matrix $A(p)$ and a linear parametric vector $b(p)$ are defined such that:

$$A(p) := \bigoplus_{i=i}^{k} p_i \otimes A_i,\ b(p) := \bigoplus_{i=i}^{k} p_i \otimes b_i,\ \ p_i \in \boldsymbol{p}_i,$$

where $A_1, \dots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \dots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$ and $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ is an interval vector of parameters.

Consider a parametric system of linear equations

$$A(p) \otimes x = b(p), \tag{3.7}$$

where $A_1, \dots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \dots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$ and $p = (p_1, \dots, p_k) \in \mathbb{R}^k(\mathcal{B}^k)$. The system (3.7) is solvable if there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $A(p) \otimes x = b(p)$, where $p \in \boldsymbol{p}$ is considered as uncertain and varying within given intervals $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$.

Note that the system (3.7) can be rewritten into the following system of equations: Suppose that $A_1 = (a_{ij}^{(1)}), \dots, A_k = (a_{ij}^{(k)}) \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \dots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$ and $p = (p_1, \dots, p_k) \in \mathbb{R}^k(\mathcal{B}^k)$ are given and define auxiliary matrices $\hat{A}$, $\hat{P}$ and $B$ as:

$$\hat{A} = \begin{pmatrix} a_{11}^{(1)} & a_{11}^{(2)} & \dots & a_{11}^{(k)} & a_{12}^{(1)} & \dots & a_{12}^{(k)} & \dots & a_{1n}^{(1)} & \dots & a_{1n}^{(k)} \\ a_{21}^{(1)} & a_{21}^{(2)} & \dots & a_{21}^{(k)} & a_{22}^{(1)} & \dots & a_{22}^{(k)} & \dots & a_{2n}^{(1)} & \dots & a_{2n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}^{(1)} & a_{n1}^{(2)} & \dots & a_{n1}^{(k)} & a_{n2}^{(1)} & \dots & a_{n2}^{(k)} & \dots & a_{nn}^{(1)} & \dots & a_{nn}^{(k)} \end{pmatrix},$$

$$\hat{P} \in \left\{ \begin{pmatrix} p_1 & \varepsilon & \varepsilon & \dots & \varepsilon \\ p_2 & \varepsilon & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_k & \varepsilon & \varepsilon & \dots & \varepsilon \\ \varepsilon & p_1 & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon & p_k & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon & \varepsilon & \dots & \varepsilon & p_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon & \varepsilon & \dots & \varepsilon & p_k \end{pmatrix}, \begin{pmatrix} p_1 & O & O & \dots & O \\ p_2 & O & O & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_k & O & O & \dots & O \\ O & p_1 & O & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & p_k & O & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \dots & O & p_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \dots & O & p_k \end{pmatrix} \right\},$$

$$B = (b_1, \dots, b_k).$$

The following lemma follows directly from the above.

**Lemma 3.1.** *$A(p) \otimes x = b(p)$ is solvable if and only if $\hat{A} \otimes \hat{P} \otimes x = B \otimes p$ is solvable.*

In the following example we will show that the set

$$\{A \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n}); (\exists p \in \boldsymbol{p}):\ A = A(p)\} \subsetneq [A(\underline{p}), A(\overline{p})],$$

i.e., there is $A \in [A(\underline{p}), A(\overline{p})]$ such that $A \neq A(p)$ holds for each $p \in \boldsymbol{p}$.

**Example 3.1.** *Consider a max-plus algebra and suppose that matrices $A_1,\ A_2,\ A_3$ and the interval vector $\boldsymbol{p} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)$ have the following forms:*

$$A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix},\ \boldsymbol{p} = \begin{pmatrix} [1,3] \\ [0,1] \\ [0,1] \end{pmatrix}.$$

*Compute the matrices $A(\underline{p}),\ A(\overline{p})$ as follows*

$$A(\underline{p}) = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix},\ A(\overline{p}) = \begin{pmatrix} 4 & 6 \\ 5 & 4 \end{pmatrix}$$

*and consider the matrix*

$$A = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \in [A(\underline{p}), A(\overline{p})].$$

*We will show that the linear system $A(p) = A$ is not solvable for each $p \in \boldsymbol{p}$:*

$$A(p) = A \Leftrightarrow p_1 \otimes \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \oplus\ p_2 \otimes \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} \oplus\ p_3 \otimes \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 5 \\ 0 & 4 & 3 \\ 1 & 3 & 2 \end{pmatrix} \otimes \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 5 \\ 3 \end{pmatrix}. \tag{3.8}$$

*Denote the linear system* (3.8) *as $C \otimes p = d$. By Theorem 2.1 the linear system* (3.8) *is not solvable since $p^*(C, d) = (2, 0, 0)^T$ is not its solution.*

The parametric solution set is defined accordingly as

$$S(A, b, \boldsymbol{p}) = \{x \in \overline{\mathbb{R}}; (\exists p \in \boldsymbol{p})\ A(p) \otimes x = b(p)\}.$$

**Lemma 3.2.** *Let $A_1, \ldots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \ldots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$ and $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ be given. Then $S(A, b, \boldsymbol{p})$ is bounded from above by the vector $x^*(A(\underline{p}), b(\overline{p}))$.*

*Proof.* Suppose that $p \in \boldsymbol{p}$ is an arbitrary vector. Then by Lemmas 2.2, 2.3 we have:

$$x^*(A(p), b(p)) \leq x^*(A(p), b(\overline{p})) \leq x^*(A(\underline{p}), b(\overline{p})).$$

□

**Lemma 3.3.** *Let $A_1, \ldots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \ldots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$ and $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ be given. If there are $x \in \mathbb{R}^n$, $p \in \boldsymbol{p}$ such that $A(p) \otimes x = b(p)$ then for any $\alpha \in \mathbb{R}$ the equation $A(\alpha \otimes p) \otimes x = b(\alpha \otimes p)$ holds.*

*Proof.* Suppose that there are $x \in \mathbb{R}^n$, $p \in \boldsymbol{p}$ such that $A(p) \otimes x = b(p)$ and $\alpha$ is an arbitrary real number. Then we have:

$$A(\alpha \otimes p) \otimes x = (\bigoplus_{i=i}^{k} \alpha \otimes p_i \otimes A_i) \otimes x = \alpha \otimes (\bigoplus_{i=i}^{k} p_i \otimes A_i) \otimes x =$$

$$\alpha \otimes \bigoplus_{i=i}^{k} p_i \otimes b_i = \alpha \otimes b(p) = b(\alpha \otimes p).$$

□

In the rest of this paper, we assume given an interval matrix $\boldsymbol{A} = [\underline{A}, \overline{A}]$ and an interval vector $\boldsymbol{p} = [\underline{p}, \overline{p}]$.

For each pair of indices $i, j \in N$, we define $\tilde{A}^{(ij)} \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$ and $\tilde{p}^{(i)} \in \mathbb{R}^k(\mathcal{B}^k)$ by putting for every $r, s \in N$ and $v \in K$

$$\widetilde{a}_{rs}^{(ij)} = \begin{cases} \overline{a}_{rs}, & \text{for } r = i,\ s = j \\ \underline{a}_{rs}, & \text{otherwise} \end{cases}, \qquad \tilde{p}_v^{(i)} = \begin{cases} \overline{p}_v, & \text{for } v = i \\ \underline{p}_v, & \text{otherwise} \end{cases}.$$

The following lemma says that every matrix in $\boldsymbol{A}$ can be written as a linear combination of *generating* matrices (in short: generators) $\tilde{A}^{(ij)}$ with $i,\ j \in N$. Similarly, every vector in $\boldsymbol{p}$ is equal to a linear combination of generators $\tilde{p}^{(i)}$ with $i \in N$.

We remind that, in the extremal algebra, $\oplus$ is the maximum operation, while $\otimes$ is either the addition $+$ or the minimum operation.

**Lemma 3.4.** [12],[19] Let $p \in \mathbb{R}^k(\mathcal{B}^k)$ and $A \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$. Then

(1) $p \in \boldsymbol{p} \subseteq \mathbb{R}^k$ if and only if $p = \bigoplus_{j=1}^{k} \alpha_j \otimes \tilde{p}^{(j)}$ for some $\alpha_j \in \mathbb{R}$ with $(\underline{p}_j - \overline{p}_j) \leq \alpha_j \leq 0$, $A \in \boldsymbol{A}$ if and only if $A = \bigoplus_{i,j\in N} \alpha_{ij} \otimes \tilde{A}^{(ij)}$ for some $\alpha_{ij} \in \mathbb{R}$ with $\underline{a}_{ij} - \overline{a}_{ij} \leq \alpha_{ij} \leq 0$ in $(\max, +)$-algebra,

(2) $p \in \boldsymbol{p} \subseteq \mathcal{B}^k$ if and only if $p = \bigoplus_{j=1}^{k} \alpha_j \otimes \tilde{p}^{(j)}$ for some $\alpha_j \in \mathcal{B}$ with $\underline{p}_j \leq \alpha_j \leq \overline{p}_j$, $A \in \boldsymbol{A}$ if and only if $A = \bigoplus_{i,j\in N} \alpha_{ij} \otimes \tilde{A}^{(ij)}$ for some $\alpha_{ij} \in \mathcal{B}$ with $\underline{a}_{ij} \leq \alpha_{ij} \leq \overline{a}_{ij}$ in $(\max, \min)$-algebra.

The following lemma presents a sufficient condition for equality $A(\boldsymbol{p}) = \boldsymbol{A}$.

**Lemma 3.5.** *Let $\boldsymbol{p} \subseteq \mathbb{R}^{n^2}(\mathcal{B}^{n^2})$, $\boldsymbol{p} = (\boldsymbol{p}_1^1, \boldsymbol{p}_2^1, \dots, \boldsymbol{p}_n^1, \boldsymbol{p}_1^2, \dots, \boldsymbol{p}_n^2, \dots, \boldsymbol{p}_1^n, \dots, \boldsymbol{p}_n^n)$ with $\boldsymbol{p}_j^i = [\underline{a}_{ij}, \overline{a}_{ij}]$ for $i, j \in N$, $\boldsymbol{A} \subseteq \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$ and $A_1^1, A_2^1, \dots, A_n^1, A_1^2, \dots, A_n^2, \dots, A_1^n, \dots, A_n^n$ with $A_j^i = \tilde{A}^{(ij)}$ for $i, j \in N$ be given. Then $A(\boldsymbol{p}) = \boldsymbol{A}$.*

*Proof.* First we prove the inclusion $\boldsymbol{A} \subseteq A(\boldsymbol{p})$. Suppose that $A \in \boldsymbol{A}$ is an arbitrary matrix. Then, in accordance with Lemma 3.4, $A = \bigoplus_{i,j\in N} \alpha_{ij} \otimes \tilde{A}^{(ij)}$. Now it suffices to put $p_j^i := \alpha_{ij}$, $A_j^i := \tilde{A}^{(ij)}$ and hence we have that $A \in A(\boldsymbol{p})$.

The reverse inclusion trivially follows. □

Observe that the inverse implication in Lemma 3.5 is not true in generally.

A parametric system of linear equations is a system of the form

$$A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p}) \tag{3.9}$$

is the set of all parametric systems of the form $A(p) \otimes x = b(p)$ for some $p \in \boldsymbol{p}$ and $\boldsymbol{p}$ is an interval vector. A parametric system of the form (3.7) is called a parametric subsystem of parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ if $p \in \boldsymbol{p}$. If we ask for the solvability of at least one of subsystems we say about the posible solvability and the universal solvability which requires solvability of all subsystems.

**Definition 3.1.** *Let $A_1,\dots,A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1,\dots,b_k \in \mathbb{R}^m(\mathcal{B}^m)$ and $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ be given.*

- *A parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is universally solvable if there is $x \in \mathbb{R}^n$ such that $A(p)\otimes x = b(p)$ holds for each $p \in \boldsymbol{p}$,*
- *A parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is possibly solvable if there are $x \in \mathbb{R}^n$ and $p \in \boldsymbol{p}$ such that $A(p)\otimes x = b(p)$ holds.*

## 4. Universally solvable parametric systems

Now we are going to present equivalent conditions for universally solvable parametric systems.

**Theorem 4.1.** *Let $A_1,\dots,A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1,\dots,b_k \in \mathbb{R}^m(\mathcal{B}^m)$ and $\boldsymbol{p} = (\boldsymbol{p}_1,\dots,\boldsymbol{p}_k)$ with $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ be given. A parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is universally solvable if and only if there is $x \in \mathbb{R}^n$ such that $A(\tilde{p}^{(i)}) \otimes x = b(\tilde{p}^{(i)})$ holds for each $i \in K$.*

*Proof.* Suppose that there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $A(\tilde{p}^{(i)}) \otimes x = b(\tilde{p}^{(i)})$ holds for each $i \in K$ and $p \in \boldsymbol{p}$ is an arbitrary vector. In accordance with Lemma 3.4 we have $p = (p_1,\dots,p_k) = \bigoplus_{j=1}^k \alpha_j \otimes \tilde{p}^{(j)}$ and $p_s = \bigoplus_{j=1}^k \alpha_j \otimes \tilde{p}^{(j)}_s$ for each $s \in K$. Then we have

$$A(p)\otimes x = (\bigoplus_{s=1}^k p_s \otimes A_s)\otimes x = (\bigoplus_{s=1}^k \bigoplus_{j=1}^k \alpha_j \otimes \tilde{p}^{(j)}_s \otimes A_s)\otimes x =$$

$$\alpha_1 \otimes \tilde{p}^{(1)}_1 \otimes A_1 \otimes x \oplus \alpha_2 \otimes \tilde{p}^{(2)}_1 \otimes A_1 \otimes x \oplus \cdots \oplus \alpha_k \otimes \tilde{p}^{(k)}_1 \otimes A_1 \otimes x \oplus$$

$$\alpha_1 \otimes \tilde{p}^{(1)}_2 \otimes A_2 \otimes x \oplus \alpha_2 \otimes \tilde{p}^{(2)}_2 \otimes A_2 \otimes x \oplus \cdots \oplus \alpha_k \otimes \tilde{p}^{(k)}_2 \otimes A_2 \otimes x \oplus$$

$$\vdots$$

$$\alpha_1 \otimes \tilde{p}^{(1)}_k \otimes A_k \otimes x \oplus \alpha_2 \otimes \tilde{p}^{(2)}_k \otimes A_k \otimes x \oplus \cdots \oplus \alpha_k \otimes \tilde{p}^{(k)}_k \otimes A_k \otimes x =$$

$$\alpha_1 \otimes (\tilde{p}^{(1)}_1 \otimes A_1 \otimes x \oplus \tilde{p}^{(1)}_2 \otimes A_2 \otimes x \oplus \cdots \oplus \tilde{p}^{(1)}_k \otimes A_k \otimes x) \oplus$$

$$\alpha_2 \otimes (\tilde{p}^{(2)}_1 \otimes A_1 \otimes x \oplus \tilde{p}^{(2)}_2 \otimes A_2 \otimes x \oplus \cdots \oplus \tilde{p}^{(2)}_k \otimes A_k \otimes x) \oplus$$

$$\vdots$$

$$\alpha_k \otimes (\tilde{p}^{(k)}_1 \otimes A_1 \otimes x \oplus \tilde{p}^{(k)}_2 \otimes A_2 \otimes x \oplus \cdots \oplus \tilde{p}^{(k)}_k \otimes A_k \otimes x) =$$

$$\alpha_1 \otimes (\bigoplus_{j=1}^k \tilde{p}^{(1)}_j \otimes A_j)\otimes x \oplus \alpha_2 \otimes (\bigoplus_{j=1}^k \tilde{p}^{(2)}_j \otimes A_j)\otimes x \oplus \cdots \oplus \alpha_k \otimes (\bigoplus_{j=1}^k \tilde{p}^{(k)}_j \otimes A_j)\otimes x =$$

$$\alpha_1 \otimes A(\tilde{p}^{(1)})\otimes x \oplus \alpha_2 \otimes A(\tilde{p}^{(2)})\otimes x \oplus \cdots \oplus \alpha_k \otimes A(\tilde{p}^{(k)})\otimes x =$$

$$\alpha_1 \otimes b(\tilde{p}^{(1)}) \oplus \alpha_2 \otimes b(\tilde{p}^{(2)}) \oplus \cdots \oplus \alpha_k \otimes b(\tilde{p}^{(k)}) =$$

$$\alpha_1 \otimes \bigoplus_{j=1}^k \tilde{p}^{(1)}_j \otimes b_j \oplus \alpha_2 \otimes \bigoplus_{j=1}^k \tilde{p}^{(2)}_j \otimes b_j \oplus \cdots \oplus \alpha_k \otimes \bigoplus_{j=1}^k \tilde{p}^{(k)}_j \otimes b_j =$$

$$\bigoplus_{s=1}^{k}\bigoplus_{j=1}^{k}\alpha_j\otimes\tilde{p}_s^{(j)}\otimes b_s=\bigoplus_{s=1}^{k}p_s\otimes b_s=b(p).$$

The reverse implication trivially follows. □

For $A_1,\dots,A_k\in\mathbb{R}^{m\times n}$, $b_1,\dots,b_k\in\mathbb{R}^n$ and $\boldsymbol{p}=(\boldsymbol{p}_1,\dots,\boldsymbol{p}_k)$ consider the linear system of equations

$$\begin{pmatrix} A(\tilde{p}^{(1)})\\ A(\tilde{p}^{(2)})\\ \vdots\\ A(\tilde{p}^{(k)})\end{pmatrix}\otimes x=\begin{pmatrix} b(\tilde{p}^{(1)})\\ b(\tilde{p}^{(2)})\\ \vdots\\ b(\tilde{p}^{(k)})\end{pmatrix}. \tag{4.10}$$

By Theorem 4.1, verification of whether a parametric system $A(\boldsymbol{p})\otimes x=b(\boldsymbol{p})$ is universally solvable reduces to verifying the non-emptyness of solution set of the linear system of equations (4.10). The problem of solvability of a linear system (4.10) can be polynomially solvable in $O(k\cdot m\cdot n)$ time (see Theorem 2.1).

## 5. Possibly solvable parametric systems

**Lemma 5.1.** *Let $A_1=(a_{ij}^{(1)}),\dots,A_k=(a_{ij}^{(k)})\in\mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $x\in\mathbb{R}^n(\mathcal{B}^n)$, $p\in\mathbb{R}^k(\mathcal{B}^k)$ be given. Then the following holds*

$$A(p)\otimes x=\begin{pmatrix} A_1\otimes x & A_2\otimes x & \dots & A_k\otimes x\end{pmatrix}\otimes p.$$

*Proof.* Suppose that $A_1,\dots,A_k\in\mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $x\in\mathbb{R}^n(\mathcal{B}^n)$, $p\in\mathbb{R}^k(\mathcal{B}^k)$ are given. Then we get

$$A(p)\otimes x=(p_1\otimes A_1\oplus\dots\oplus p_k\otimes A_k)\otimes x=$$

$$\begin{pmatrix} p_1\otimes a_{11}^{(1)} & \dots & p_1\otimes a_{n1}^{(1)}\\ \vdots & \vdots & \vdots\\ p_1\otimes a_{n1}^{(1)} & \dots & p_1\otimes a_{n1}^{(1)}\end{pmatrix}\otimes x\oplus\dots\oplus\begin{pmatrix} p_k\otimes a_{11}^{(k)} & \dots & p_k\otimes a_{n1}^{(k)}\\ \vdots & \vdots & \vdots\\ p_k\otimes a_{n1}^{(k)} & \dots & p_k\otimes a_{n1}^{(k)}\end{pmatrix}\otimes x=$$

$$\begin{pmatrix} (p_1\otimes a_{11}^{(1)}\oplus\dots\oplus p_k\otimes a_{11}^{(k)})\otimes x_1\oplus & \dots & \oplus(p_1\otimes a_{1n}^{(1)}\oplus\dots\oplus p_k\otimes a_{1n}^{(k)})\otimes x_n\\ \vdots & \vdots & \vdots\\ (p_1\otimes a_{n1}^{(1)}\oplus\dots\oplus p_k\otimes a_{n1}^{(k)})\otimes x_1\oplus & \dots & \oplus(p_1\otimes a_{nn}^{(1)}\oplus\dots\oplus p_k\otimes a_{nn}^{(k)})\otimes p_n\end{pmatrix}=$$

$$\begin{pmatrix} (a_{11}^{(1)}\otimes x_1\oplus\dots\oplus a_{1n}^{(1)}\otimes x_n)\otimes p_1\oplus & \dots & \oplus(a_{11}^{(k)}\otimes x_1\oplus\dots\oplus a_{1n}^{(k)}\otimes x_n)\otimes p_k\\ \vdots & \vdots & \vdots\\ (a_{n1}^{(1)}\otimes x_1\oplus\dots\oplus a_{nn}^{(1)}\otimes x_n)\otimes p_1\oplus & \dots & \oplus(a_{n1}^{(k)}\otimes x_1\oplus\dots\oplus a_{nn}^{(k)}\otimes x_n)\otimes p_k\end{pmatrix}=$$

$$\begin{pmatrix} (a_{11}^{(1)}\otimes x_1\oplus\dots\oplus a_{1n}^{(1)}\otimes x_n) & \dots & (a_{11}^{(k)}\otimes x_1\oplus\dots\oplus a_{1n}^{(k)}\otimes x_n)\\ \vdots & \vdots & \vdots\\ (a_{n1}^{(1)}\otimes x_1\dots\oplus a_{nn}^{(1)}\otimes x_n) & \dots & (a_{n1}^{(k)}\otimes x_1\oplus\dots\oplus a_{nn}^{(k)}\otimes x_n)\end{pmatrix}\otimes p=$$

$$\begin{pmatrix} A_1\otimes x & A_2\otimes x & \dots & A_k\otimes x\end{pmatrix}\otimes p.$$

□

**Theorem 5.1.** *Let $A_1, \dots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \dots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$, $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ be given. If a parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is possibly solvable then*

$$A(\overline{p}) \otimes x^*(A(\underline{p}), b(\overline{p})) \geq b(\underline{p}).$$

*Proof.* Suppose that a parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is possibly solvable then there are $x \in \mathbb{R}^n$, $p \in \boldsymbol{p}$ such that $A(p) \otimes x = b(p)$. Using monotonicity and Lemma 2.2, Lemma 2.3 we get

$$A(\overline{p}) \otimes x^*(A(\underline{p}), b(\overline{p})) \geq A(p) \otimes x^*(A(\underline{p}), b(\overline{p})) \geq$$

$$A(p) \otimes x^*(A(p), b(p)) \geq A(p) \otimes x = b(p) \geq b(\underline{p}).$$

□

**Theorem 5.2.** *Let $A_1, \dots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \dots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$, $B = (b_1 \dots b_k) \in \mathbb{R}^{m\times k}(\mathcal{B}^{m\times k})$ and $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ with $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ be given. If the linear system*

$$\begin{aligned} A(p) \otimes x^*(A(\underline{p}), b(\overline{p})) &= b(p) \\ A(p) \otimes x^*(A(\overline{p}), b(\underline{p})) &= b(p) \end{aligned} \tag{5.11}$$

*is solvable then the parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is possibly solvable.*

*Proof.* Suppose that $p \in \boldsymbol{p}$ is a solution of (5.11). By Lemma 2.2 and Lemma 2.3 the inequalities

$$x^*(A(\overline{p}), b(\underline{p})) \leq x^*(A(p), b(p)) \leq x^*(A(\underline{p}), b(\overline{p}))$$

hold for each $p \in \boldsymbol{p}$ and after multiplying them by $A(p)$ we obtain

$$A(p) \otimes x^*(A(\overline{p}), b(\underline{p})) \leq A(p) \otimes x^*(A(p), b(p)) \leq A(p) \otimes x^*(A(\underline{p}), b(\overline{p})).$$

Then by (5.11) we obtain

$$b(p) = A(p) \otimes x^*(A(\overline{p}), b(\underline{p})) \leq A(p) \otimes x^*(A(p), b(p)) \leq A(p) \otimes x^*(A(\underline{p}), b(\overline{p})) = b(p)$$

which implies

$$b(p) = A(p) \otimes x^*(A(p), b(p))$$

and hence the parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is possibly solvable. □

Observe that (5.11) can be equivalently rewritten into two sided linear system as follows:

$$\begin{pmatrix} A_1 \otimes x^*(A(\underline{p}), b(\overline{p})) & \oplus & \dots & \oplus & A_k \otimes x^*(A(\underline{p}), b(\overline{p})) \\ A_1 \otimes x^*(A(\overline{p}), b(\underline{p})) & \oplus & \dots & \oplus & A_k \otimes x^*(A(\overline{p}), b(\underline{p})) \end{pmatrix} \otimes p = \begin{pmatrix} B \\ B \end{pmatrix} \otimes p \tag{5.12}$$

Notice that Theorem 5.2 provides a sufficient condition for a possible solvability of $A(\boldsymbol{p})\otimes x = b(\boldsymbol{p})$. Thus, if the two sided linear system (5.12) is solvable then $A(\boldsymbol{p})\otimes x = b(\boldsymbol{p})$ is possibly solvable, i.e. we have to verify the non-emptyness of solution set of the system of max-plus (max-min) equations (5.12). Algebraically, we solve two sided system $C\otimes x = D \otimes x$. Notice that it follows from the results in [2] that max-plus two sided systems are polynomially equivalent to mean payoff games which is belonging to the class $NP \cap co$-$NP$ which is in the contrast with a solvability of max-min two sided systems for which there is a polynomial algorithm, see [13].

Moreover, Theorem 5.2 allows us to design an iterative process that can lead to a solution to (5.11). If (5.11) is not solvable, we need to slightly modify the entries $\underline{p}$, $\overline{p}$ (increase $\underline{p}$ a little and decrease $\overline{p}$ a little) and repeat the whole process.

## 6. Other versions of solvability

Suppose that $\mathcal{A}$, $\mathcal{E} \subseteq K$, $\mathcal{A} \cup \mathcal{E} = K$, $\mathcal{A} \cap \mathcal{E} = \emptyset$, $A_1, \dots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \dots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$, $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ with $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ are given. Moreover, without loss of generality, suppose that $\mathcal{A} = \{1, \dots, r\}$ and $\mathcal{E} = \{r+1, \dots, k\}$.

**Definition 6.1.** *Let $A_1, \dots, A_k \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_1, \dots, b_k \in \mathbb{R}^n(\mathcal{B}^m)$ and $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ be given. A parametric system $A(\boldsymbol{p}_\mathcal{A}) \otimes x = b(\boldsymbol{p}_\mathcal{E})$ is*

- *tolerably EA-solvable if there are $x \in \mathbb{R}^n$, $p_\mathcal{E} \in \boldsymbol{p}_\mathcal{E}$ such that $A(p_\mathcal{A}) \otimes x = b(p_\mathcal{E})$ holds for each $p_\mathcal{A} \in \boldsymbol{p}_\mathcal{A}$,*
- *tolerably AE-solvable if there is $x \in \mathbb{R}^n$ such that for each $p_\mathcal{A} \in \boldsymbol{p}_\mathcal{A}$ there is $p_\mathcal{E} \in \boldsymbol{p}_\mathcal{E}$ such that $A(p_\mathcal{A}) \otimes x = b(p_\mathcal{E})$,*
- *controllable AE-solvable if there is $x \in \mathbb{R}^n$ such that for each $p_\mathcal{E} \in \boldsymbol{p}_\mathcal{E}$ there is $p_\mathcal{A} \in \boldsymbol{p}_\mathcal{A}$ such that $A(p_\mathcal{A}) \otimes x = b(p_\mathcal{E})$,*
- *weakly EE-solvable if there are $x \in \mathbb{R}^n$, $p_\mathcal{E} \in \boldsymbol{p}_\mathcal{E}$ and $p_\mathcal{A} \in \boldsymbol{p}_\mathcal{A}$ such that $A(p_\mathcal{A}) \otimes x = b(p_\mathcal{E})$.*

**Theorem 6.1.** *A parametric system $A(\boldsymbol{p}_\mathcal{A}) \otimes x = b(\boldsymbol{p}_\mathcal{E})$ is tolerably EA-solvable if and only if there are $x \in \mathbb{R}^n(\mathcal{B}^n)$ and $p_\mathcal{E} \in \boldsymbol{p}_\mathcal{E}$ such that*

$$A(\underline{p}_\mathcal{A}) \otimes x = A(\overline{p}_\mathcal{A}) \otimes x = b(p_\mathcal{E}).$$

*Proof.* Suppose that there are $x \in \mathbb{R}^n$ and $p_\mathcal{E} \in \boldsymbol{p}_\mathcal{E}$ such that $A(\underline{p}_\mathcal{A}) \otimes x = A(\overline{p}_\mathcal{A}) \otimes x = b(p_\mathcal{E})$. Then for an arbitrary $p_\mathcal{A} \in \boldsymbol{p}_\mathcal{A}$ we have

$$b(p_\mathcal{E}) = A(\underline{p}_\mathcal{A}) \otimes x \leq A(p_\mathcal{A}) \otimes x \leq A(\overline{p}_\mathcal{A}) \otimes x = b(p_\mathcal{E}).$$

The reverse implication trivially follows. □

For $A_1, \dots, A_r \in \mathbb{R}^{m\times n}$, $b_{r+1}, \dots, b_k \in \mathbb{R}^n$, $\tilde{B} = \{b_{r+1} \dots b_k\}$ and $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ consider the system of equations

$$\begin{pmatrix} A(\underline{p}_\mathcal{A}) \\ A(\overline{p}_\mathcal{A}) \end{pmatrix} \otimes x = \begin{pmatrix} \tilde{B} \\ \tilde{B} \end{pmatrix} \otimes p_\mathcal{E}. \tag{6.13}$$

Note that similarly as above, by Theorem 6.1, verification of whether a parametric system $A(\boldsymbol{p}_\mathcal{A}) \otimes x = b(\boldsymbol{p}_\mathcal{E})$ is tolerably EA-solvable reduces to verifying two-sided linear system of equations (6.13).

Consider now the special case when $\mathcal{A} = \{1, \dots, k-1\}$ and $\mathcal{E} = \{k\}$. For $A_1, \dots, A_{k-1} \in \mathbb{R}^{m\times n}$, $b_k \in \mathbb{R}^n$ and $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ consider the system of max-plus equations

$$\begin{pmatrix} A(\underline{p}_\mathcal{A}) \\ A(\overline{p}_\mathcal{A}) \end{pmatrix} \otimes x = \begin{pmatrix} b_k \\ b_k \end{pmatrix} \otimes p_k. \tag{6.14}$$

Observe that in this case by Theorem 6.1, verification of whether a parametric system $A(\boldsymbol{p}) \otimes x = b(\boldsymbol{p})$ is tolerably EA-solvable reduces to verifying the non-emptyness of solution set of the system of max-plus/min equations (6.14). Algebraically, we have to solve parametric system $C \otimes x = \lambda \otimes y$.

**Theorem 6.2.** [11] *Suppose given $C \in \mathbb{R}^{r\times t}$, $b \in \mathbb{R}^r$ and $\underline{y}, \overline{y} \in \mathbb{R}^t$. The problem of recognizing the solvability of bounded parametric max-plus linear system $C \otimes y = \lambda \otimes b$ with bounds $\underline{y} \leq y \leq \overline{y}$, for some value of parameter $\lambda \in \mathbb{R}$, can be solved in $O(rt)$ time.*

**Corollary 6.1.** *Suppose that $A_1, \dots, A_{k-1} \in \mathbb{R}^{m\times n}$, $b_k \in \mathbb{R}^m$, $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ with $\boldsymbol{p}_i = [\underline{p}_i, \overline{p}_i]$ are given. The recognition problem whether a parametric system $A(\boldsymbol{p}_\mathcal{A}) \otimes x = b(\boldsymbol{p}_\mathcal{E})$ is tolerably EA-solvable can be checked in $O(n^2)$ time.*

*Proof.* The assertion follows from Theorem 6.2 and the fact that the dimension of the matrix on the left-hand side of (6.14) is $2n \times n$. □

For $\mathcal{A} = \{1, \ldots, r\}$, $\mathcal{E} = \{r+1, \ldots, k\}$, $A_1, \ldots, A_r \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$ and $b_{r+1}, \ldots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$ define its interval hulls by

$$\mathcal{H}_{\mathcal{A}} := \{A \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n});\ \exists p_{\mathcal{A}} = (p_1, \ldots, p_r) \in \boldsymbol{p}_{\mathcal{A}}:\ A = \bigoplus_{i=1}^{r} p_i \otimes A_i\},$$

$$\mathcal{H}_{\mathcal{E}} := \{b \in \mathbb{R}^m(\mathcal{B}^m);\ \ \exists p_{\mathcal{E}} = (p_{r+1}, \ldots, p_k) \in \boldsymbol{p}_{\mathcal{E}} : b = \bigoplus_{i=r+1}^{k} p_i \otimes b_i\}$$

and for fixed $x \in \mathbb{R}^n(\mathcal{B}^n)$ denote

$$\mathcal{H}_{\mathcal{A}} \otimes x := \{A \otimes x \in \mathbb{R}^m(\mathcal{B}^m);\ A \in \mathcal{H}_{\mathcal{A}}\}.$$

Observe that

$$\mathcal{H}_{\mathcal{A}} \subseteq [A(\underline{p}_{\mathcal{A}}), A(\overline{p}_{\mathcal{A}})],\ \mathcal{H}_{\mathcal{E}} \subseteq [b(\underline{p}_{\mathcal{E}}), b(\overline{p}_{\mathcal{E}})],\ \mathcal{H}_{\mathcal{A}} \otimes x \subseteq [A(\underline{p}_{\mathcal{A}}) \otimes x, A(\overline{p}_{\mathcal{A}}) \otimes x].$$

**Theorem 6.3.** *Let $A_1, \ldots, A_r \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_{r+1}, \ldots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$, $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)$ be given. A parametric system $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is*

(1) *tolerably AE-solvable if and only if there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $\mathcal{H}_{\mathcal{A}} \otimes x \subseteq \mathcal{H}_{\mathcal{E}}$,*
(2) *controllable AE-solvable if and only if there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $\mathcal{H}_{\mathcal{E}} \subseteq \mathcal{H}_{\mathcal{A}} \otimes x$,*
(3) *weakly EE-solvable if and only if there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $\mathcal{H}_{\mathcal{A}} \otimes x \cap \mathcal{H}_{\mathcal{E}} \neq \emptyset$.*

*Proof.* All implications will be proven indirectly and both implications will be merged into one equivalence, i.e. instead of the equivalence $A \Leftrightarrow B$ we will prove $B' \Leftrightarrow A'$. Then we get

(1) $(\forall x)\, \mathcal{H}_{\mathcal{A}} \otimes x \not\subseteq \mathcal{H}_{\mathcal{E}} \Leftrightarrow (\forall x)\, (\exists p_{\mathcal{A}})\, A(p_{\mathcal{A}}) \otimes x \notin \mathcal{H}_{\mathcal{E}} \Leftrightarrow (\forall x)(\exists p_{\mathcal{A}})(\forall p_{\mathcal{E}})\, A(p_{\mathcal{A}}) \otimes x \neq b(p_{\mathcal{E}}) \Leftrightarrow$ a parametric system $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is not tolerably AE-solvable.
(2) $(\forall x)\, \mathcal{H}_{\mathcal{E}} \not\subseteq \mathcal{H}_{\mathcal{A}} \otimes x \Leftrightarrow (\forall x)\, (\exists p_{\mathcal{E}})\, b(p_{\mathcal{E}}) \notin \mathcal{H}_{\mathcal{A}} \otimes x \Leftrightarrow (\forall x)(\exists p_{\mathcal{E}})(\forall p_{\mathcal{A}})\, A(p_{\mathcal{A}}) \otimes x \neq b(p_{\mathcal{E}}) \Leftrightarrow$ a parametric system $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is not controllable AE-solvable.
(3) $(\forall x)\, \mathcal{H}_{\mathcal{E}} \cap \mathcal{H}_{\mathcal{A}} \otimes x = \emptyset \Leftrightarrow (\forall x)(\forall p_{\mathcal{E}})(\forall p_{\mathcal{A}})\, A(p_{\mathcal{A}}) \otimes x \neq b(p_{\mathcal{E}}) \Leftrightarrow$ a parametric system $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is not weakly EE-solvable.

□

**Theorem 6.4.** *Let $A_1, \ldots, A_r \in \mathbb{R}^{m\times n}(\mathcal{B}^{m\times n})$, $b_{r+1}, \ldots, b_k \in \mathbb{R}^m(\mathcal{B}^m)$, $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)$ be given. If a parametric system $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is*

(1) *tolerably AE-solvable then $A(\underline{p}_{\mathcal{A}}) \otimes x^*(A(\overline{p}_{\mathcal{A}}), b(\overline{p}_{\mathcal{E}})) \geq b(\underline{p}_{\mathcal{E}})$,*
(2) *controllable AE-solvable then $A(\overline{p}_{\mathcal{A}}) \otimes x^*(A(\underline{p}_{\mathcal{A}}), b(\underline{p}_{\mathcal{E}})) \geq b(\overline{p}_{\mathcal{E}})$,*
(3) *weakly EE-solvable then $A(\overline{p}_{\mathcal{A}}) \otimes x^*(A(\underline{p}_{\mathcal{A}}), b(\overline{p}_{\mathcal{E}})) \geq b(\underline{p}_{\mathcal{E}})$.*

*Proof.* 1. If $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is tolerably AE-solvable then, by Theorem 6.3, there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $\mathcal{H}_{\mathcal{A}} \otimes x \subseteq \mathcal{H}_{\mathcal{E}}$. This implies that

$$A(\underline{p}_{\mathcal{A}}) \otimes x \geq b(\underline{p}_{\mathcal{E}}),$$
$$A(\overline{p}_{\mathcal{A}}) \otimes x \leq b(\overline{p}_{\mathcal{E}})$$

and hence, by Theorem 2.2, we get $A(\underline{p}_{\mathcal{A}}) \otimes x^*(A(\overline{p}_{\mathcal{A}}), b(\overline{p}_{\mathcal{E}})) \geq b(\underline{p}_{\mathcal{E}})$.

2. If $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is controllable AE-solvable then, by Theorem 6.3, there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $\mathcal{H}_{\mathcal{E}} \subseteq \mathcal{H}_{\mathcal{A}} \otimes x$. This implies that

$$A(\underline{p}_{\mathcal{A}}) \otimes x \leq b(\underline{p}_{\mathcal{E}}),$$
$$A(\overline{p}_{\mathcal{A}}) \otimes x \geq b(\overline{p}_{\mathcal{E}})$$

and hence, by Theorem 2.2, we get $A(\overline{p}_{\mathcal{A}}) \otimes x^*(A(\underline{p}_{\mathcal{A}}), b(\underline{p}_{\mathcal{E}})) \geq b(\overline{p}_{\mathcal{E}})$.

3. If $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is weakly EE-solvable then, by Theorem 6.3, there is $x \in \mathbb{R}^n(\mathcal{B}^n)$ such that $\mathcal{H}_{\mathcal{E}} \cap \mathcal{H}_{\mathcal{A}} \otimes x \neq \emptyset$. This implies that

$$\begin{aligned} A(\underline{p}_{\mathcal{A}}) \otimes x &\leq b(\overline{p}_{\mathcal{E}}) \\ A(\overline{p}_{\mathcal{A}}) \otimes x &\geq b(\underline{p}_{\mathcal{E}}) \end{aligned}$$

and hence, by Theorem 2.2, we get $A(\overline{p}_{\mathcal{A}}) \otimes x^*(A(\underline{p}_{\mathcal{A}}), b(\overline{p}_{\mathcal{E}})) \geq b(\underline{p}_{\mathcal{E}})$. □

The question that arises is whether the above necessary conditions are also sufficient conditions for the types of solvability considered. In the following example we demonstrate that it is not valid for tolerable AE-solvability.

**Example 6.1.** In max-plus algebra, decide whether a given interval parametric system $A(\boldsymbol{p}_{\mathcal{A}}) \otimes x = b(\boldsymbol{p}_{\mathcal{E}})$ is tolerably AE-solvable, if $\mathcal{A} = \{1\}$, $\mathcal{E} = \{2, 3, 4\}$ and

$$A_1 = \begin{pmatrix} 5 & 8 & 1 \\ 8 & 4 & 2 \\ 9 & 6 & 2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} [0,0] \\ [1,2] \\ [0,1] \\ [0,1] \end{pmatrix}.$$

We have

$$\mathcal{H}_{\mathcal{A}} := \{A_1\}, \qquad \mathcal{H}_{\mathcal{E}} := \{b;\ b = p_2 \otimes b_2 \oplus p_3 \otimes b_3 \oplus p_4 \otimes b_4; (p_2, p_3, p_4) \in (\boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4)\}$$

Since $\mathcal{H}_A$ consists from the only matrix $A_1$, we shall write $A$ instead of $A_1$.

First, we check whether the necessary condition for tolerance AE-solvability, given in Theorem 6.4, holds. We have $A(\underline{p}_{\mathcal{A}}) = A(\overline{p}_{\mathcal{A}}) = A$, $b(\underline{p}_{\mathcal{E}}) = (5, 4, 3)^\top$, $b(\overline{p}_{\mathcal{E}}) = (6, 5, 4)^\top$.

Since $A \otimes x^*(A, b(\overline{p}_{\mathcal{E}})) = A \otimes (-5, -2, 2)^\top = (6, 4, 4)^\top \geq b(\underline{p}_{\mathcal{E}})$ the necessary condition for tolerance AE-solvability holds.

We will show that, nevertheless, the given parametric system is not tolerance AE-solvable, i. e., for each $x \in \mathbb{R}^n$ there exists $p_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}}$ such that $A(p_{\mathcal{A}}) \otimes x \neq b(p_{\mathcal{E}})$ for each $p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}$.

In our example, we have to prove that for each $x \in \mathbb{R}^n$ and for each $p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}$ the inequality $A \otimes x \neq b(p_{\mathcal{E}})$ holds. In accordance with Theorem 2.1, it is equivalent to that $A \otimes x^*(A, b(p_{\mathcal{E}})) \neq b(p_{\mathcal{E}})$ for each $p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}$.

We show that, despite the fact that the set $\mathcal{H}_{\mathcal{E}}$ contains infinitely many elements, it is sufficient to consider integer values of the parameter $p_{\mathcal{E}}$. Considering all entries in $p_{\mathcal{E}}$, we obtain the set

$$\mathcal{H}_{\mathcal{E}}^{int} = \{(6, 5, 4)^\top,\ (5, 4, 3)^\top,\ (6, 4, 3)^\top,\ (5, 5, 4)^\top\} \subsetneq \mathcal{H}_{\mathcal{E}}.$$

We have already shown that $A \otimes x^*(A, \overline{p}_{\mathcal{E}}) = (6, 4, 4)^\top \neq \overline{p}_{\mathcal{E}}$. It can be easily verified that $A \otimes x^*(A, b(p_{\mathcal{E}})) \neq b(p_{\mathcal{E}})$ for each $p_{\mathcal{E}} \in \mathcal{H}_{\mathcal{E}}^{int}$.

Let $p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}$ be such that $b(p_{\mathcal{E}}) \in \mathcal{H}_{\mathcal{E}}$ and $b(p_{\mathcal{E}}) \notin \mathcal{H}_{\mathcal{E}}^{int}$. Then, there exist $\tilde{b} \in \mathcal{H}_{\mathcal{E}}^{int}$ and $\delta_1, \delta_2, \delta_3$, $0 \leq \delta_i < 1$, such that $b(p_{\mathcal{E}})_i = \tilde{b}_i + \delta_i$ for $i = 1, 2, 3$. Since $A \otimes x^*(A, \tilde{b}) \neq \tilde{b}$, in accordance with Theorem 2.1 we obtain that $\bigcup_{j=1}^{3} M_j(A, \tilde{b}) \neq M$. We prove that $\bigcup_{j=1}^{3} M_j(A, b(p_{\mathcal{E}})) \subseteq \bigcup_{j=1}^{3} M_j(A, \tilde{b})$. Let $i \in M$ be such that $i \notin \bigcup_{j=1}^{3} M_j(A, \tilde{b})$, i. e., for each $j \in N$, $i \notin M_j(A, \tilde{b})$. Let $j \in N$ be arbitrary but fixed. We prove that $i \notin M_j(A, b(p_{\mathcal{E}}))$.

Let $k \in M$ be such that $k \in M_j(A, \tilde{b})$. Since all elements in $A$ and $\tilde{b}$ are integers, we obtain $(\tilde{b}_i - a_{ij}) - (\tilde{b}_k - a_{kj}) \geq 1$. Then

$$(b(p_{\mathcal{E}})_i - a_{ij}) - (b(p_{\mathcal{E}})_k - a_{kj}) = (\tilde{b}_i + \delta_i - a_{ij}) - (\tilde{b}_k + \delta_k - a_{kj}) =$$
$$(\tilde{b}_i - a_{ij}) - (\tilde{b}_k - a_{kj}) + \delta_i - \delta_k \geq 1 + \delta_i - \delta_k > 0,$$

where the last inequality follows from $\delta_i - \delta_k > -1$.

We get $(b(p_{\mathcal{E}})_i - a_{ij}) > (b(p_{\mathcal{E}})_k - a_{kj})$ and therefore $i \notin M_j(A, b(p_{\mathcal{E}}))$. Then $M_j(A, b(p_{\mathcal{E}})) \subseteq M_j(A, \tilde{b})$ and consequently $\bigcup_{j=1}^{3} M_j(A, b(p_{\mathcal{E}})) \subseteq \bigcup_{j=1}^{3} M_j(A, \tilde{b})$. Since $\bigcup_{j=1}^{3} M_j(A, b(p_{\mathcal{E}})) \neq M$, by Theorem 2.1 the equation $A \otimes x = b(p_{\mathcal{E}})$ is not solvable.
From the above it follows that the given parametric system is not tolerance AE-solvable.

In our opinion, there are similar counterexamples for the other two types of solvability, so that Theorem 6.4 provides necessary but not sufficient conditions for the above solvability concepts.

## 7. Conclusion

In this paper, the notions of max-plus and max-min linear parametric systems and their solvability were introduces. In addition, several types of linear parametric systems were defined, based on splitting an interval parametric vector $\boldsymbol{p}$ into two vectors according to forall–exists quantification of its interval entries. Equivalent conditions for these concepts of solvability, namely tolerable EA/AE-solvability, controlable AE-solvability and weakly EE-solvability, were presented and also polynomially checking sufficient conditions were given. For further research we propose the following topics:

Suggest polynomial algorithm for checking possible solvability or to prove NP-hardness of the problem.

Find efficient equivalent conditions for tolerable EA/AE-solvability, controlable AE-solvability and weakly EE-solvability of parametric linear systems.

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