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A fixed point theorem revisited

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ABSTRACT. One of the main results in the fixed point structure theory is the following [I. A. Rus, *Fixed Point Structure Theory*, Cluj University Press, Cluj-Napoca, 2006]:

Theorem 0.1 (Fixed Point Principle of $(\theta, \varphi)-$contractions, Rus [70]**).** *Let $(X, S(X), M)$ be a fixed pint structure on a set X with structure and (θ, η) a compatible pair with $(X, S(X), M)$, where $\theta : Z \subset P(X) \to \mathbb{R}_+$, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure mapping and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We assume that:*

(1) $\theta|_{\eta(Z)}$ has the intersection property;
(2) f is a $(\theta, \varphi)-$contraction.

Then:

(i) $I(f) \cap S(X) \neq \emptyset$, i.e., f has an invariant subset in $S(X)$;
(ii) $F_f \neq \emptyset$, i.e., there exists $x^* \in Y$ such that $f(x^*) = x^*$;
(iii) if $F_f \in Z$, then $\theta(F_f) = 0$.

The aim of this paper is fourfold:

(a) to explain the concepts which appear in this theorem;
(b) to present a heuristic introduction to the theorem;
(c) to present some relevant corollaries of the theorem;
(d) to regard the theorem as a source of new fixed point results.

1. INTRODUCTION

The structure of the remaining part of the paper is the following:

2. Notations
3. Fixed point structure on a set with structure
4. Closure mappings
5. Functional with the intersection property
6. Compatible pair with a fixed point structure
7. The problem of the invariant fixed point space
8. $(\theta, \varphi)-$contractions
9. Fixed point principle of $(\theta, \varphi)-$contractions
10. Corollaries and the new possible results

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2. NOTATIONS

Let X be a set and $f : X \to X$ a mapping. We denote:

$\mathcal{P}(X) = \{A | A \subset X\}$;
$P(X) = \{A \subset X | A \neq \emptyset\}$;
$F_f = \{x \in X | f(x) = x\}$, the fixed point set of f;
$I(f) = \{A \in P(X) | f(A) \subset A\}$, the nonempty invariant subsets of f.

Let (X, τ) be a topological space. In this setting we shall use the notations:

$P_{cl}(X) = \{A \in P(X) | A = \overline{A}$, the set of all nonempty closed subsets of $X\}$;
$P_{cp}(X) = \{A \in P(X) |$A is compact$\}$;
$P_{cn}(X) = \{A \in P(X) |$A is connex$\}$.

Let $(X, +, \mathbb{R})$ be a real linear space. In this setting we shall use the notations:

$P_{cv}(X) = \{A \in P(X) |$A is convex$\}$;
$P_{st}(X) = \{A \in P(X) |$A is star shaped$\}$.

Let (X, d) be a metric space. We denote:
$B(a, r) := \{x \in X | d(x, a) < r\}$ (the open ball centered at a);
$\overline{B}(a, r) := \{x \in X | d(x, a) \leq r\}$ (the closed ball centered at a).
Then we have $P_{cl}(X)$, $P_{cp}(X)$, $P_{cn}(X)$ considered with respect to (X, τ_d) and

$$P_b(X) := \{A \in P(X) | \delta(A) < +\infty\}, \text{ where } \delta(A) \text{ is the diameter of A.}$$

Let A, B be two non-empty sets. Then:

$$\mathbb{M}(A, B) := \{f : A \to B | f = \text{ a mapping}\}$$
$$\mathbb{M}(A) := \mathbb{M}(A, A).$$

For more considerations on the above notions, see Rus [54], [67], [68], [69], [76], [70], [71], [77], [80], ...

3. FIXED POINT STRUCTURE ON A SET WITH STRUCTURE

Let X be a nonempty set with structure. We assume that for each $A \in P(X)$ is given a subset $M(A) \subset \mathbb{M}(A)$. Then we denote:

$$T = \{f : A \to A | f \in M(A), A \in P(X)\}.$$

So $f \in T$ is equivalent to the fact that there exists $A \in P(X)$ such that $f \in M(A)$.

Definition 3.1 (Rus [61], [62], [63]). *A triple $(X, S(X), M)$ is said to be* a fixed point structure *on X if and only if*

(1) $S(X) \subset P(X)$, $S(X) \neq \emptyset$;
(2) $A \in P(X)$, $f \in M(A)$, $B \in P(A) \Rightarrow (f|_B : B \to B) \in M(B)$;
(3) $A \in S(X)$, $f \in M(A) \Rightarrow F_f \neq \emptyset$, i.e., A has the fixed point property with respect to $M(A)$.

An element of $S(X)$ is called a fixed point space.

Example 3.1 (The trivial fixed point structure). *X:= a nonempty set, $S(X) := \{\{x\} | x \in X\}$ and $M(A) := \mathbb{M}(A)$.*

Example 3.2 (The fixed point structure of contractions). *Let X be a nonempty set endowed with a metric d such that (X, d) is a complete metric space, take $S(X) := P_{cl}(X)$ and for any $A \in P(X)$ define*

$$M(A) := \{f : A \to A | d(f(x), f(y)) \leq \alpha d(x, y), \forall x, y \in A\},$$

where $0 \leq \alpha < 1$.

Then, by the contraction mapping principle, see for example Rus and Petruşel [2] [77], *$(X, S(X), M)$ is a fixed point structure.*

Example 3.3 (The fixed point structure of almost contractions). *Let X be a nonempty set endowed with a metric d such that (X, d) is a complete metric space. Take $S(X) := P_{cl}(X)$ and for any $A \in P(X)$ define*

$$M(A) := \{f : A \to A | d(f(x), f(y)) \leq \alpha d(x, y) + Ld(y, f(x)), \forall x, y \in A\},$$

where $0 \leq \alpha < 1$ and $L \geq 0$.

Then, by the almost contraction mapping principle, see Berinde [10], [11], *$(X, S(X), M)$ is a fixed point structure.*

Example 3.4 (The fixed point structure of contractive mappings). *Let (X, d) be a complete metric space, $S(X) := P_{cp}(X)$ and for any $A \in P(X)$ define*

$$M(A) := \{f : A \to A | d(f(x), f(y)) < d(x, y), \forall x, y \in A, x \neq y\}.$$

Then, by Nemytzki-Edelstein fixed point theorem, see Nemytsky [46] *(1936), and Edelstein* [27] *(1962), $(X, S(X), M)$ is a fixed point structure.*

Example 3.5 (The fixed point structure of enriched contractions). *Let X be a Banach space. Take $S(X) := P_{cl,cv}(X)$ and for any $A \in P(X)$ define*

$$M(A) := \{f : A \to A | \|b(x - y) + f(x) - f(y)\| \leq a\|x - y\|, \forall x, y \in A\},$$

where $0 \leq a < 1$ and $b \geq 0$.

Then, by the enriched contractions fixed point theorem, see Berinde and Păcurar [17], *$(X, S(X), M)$ is a fixed point structure.*

Example 3.6 (The first fixed point structure of Schauder). *Let X be a Banach space, $S(X) := P_{cp,cv}(X)$ and for $A \in P(X)$ define*

$$M(A) = C(A, A) := \{f : A \to A | f \text{ is continuous}\}.$$

Then, by the first fixed point theorem of Schauder Schauder [83], *$(X, S(X), M)$ is a fixed point structure.*

Example 3.7 (The second fixed point structure of Schauder). *Let X be a Banach space, $S(X) := P_{b,cp,cv}(X)$ and for $A \in P(X)$ define*

$$M(A) := \{f : A \to A | f \text{ is completely continuous}\}.$$

Then, by the second fixed point theorem of Schauder, see for example Morris and Noussair [44], *$(X, S(X), M)$ is a fixed point structure.*

Example 3.8 (The fixed point structure of Dotson). *Let X be a Banach space, $S(X) := P_{cp,st}(X)$ and for $A \in P(X)$ define*

$$M(A) := \{f : A \to A | f \text{ is nonexpansive}\}.$$

Then, by a fixed point theorem of Dotson [26], *$(X, S(X), M)$ is a fixed point structure.*

Example 3.9 (The fixed point structure of Browder). *Let X be a Hilbert space, $S(X) := P_{b,cp,cv}(X)$ and for $A \in P(X)$ define*

$$M(A) := \{f : A \to A | f \text{ is nonexpansive}\}.$$

Then, by a fixed point theorem of Browder [21], *$(X, S(X), M)$ is a fixed point structure.*

Example 3.10 (The fixed point structure of enriched nonexpansive mappings). *Let X be a Hilbert space, $S(X) := P_{b,cl,cv}(X)$ and for $A \in P(X)$ define*

$$M(A) := \{f : A \to A | f \text{ is demicompact}$$
$$\text{and } \|b(x-y) + f(x) - f(y)\| \leq (b+1)\|x-y\|\},$$

where $b \geq 0$.

Then, by a fixed point theorem in Berinde [14], *$(X, S(X), M)$ is a fixed point structure.*

Remark 3.1. *For other examples of fixed point structures, see Rus* [70], [63], [73], [74],

For the fixed point theorems which generate the above fixed point structures, see Granas and Dugundji [32], *Rus et al.* [77], *Berinde* [11], *Rus and Şerban* [79], *Deimling* [25], *Rus* [69], *Krasnoselskii and Zabrejko* [37].

Remark 3.2. *Other examples of fixed point structures could be adapted to Definition 3.1 from the papers Muenzenberger* [42], *and Muenzenberger and Smithson* [43], *where the authors considered similar axiomatic fixed point structures which permitted the deduction of a number of known fixed point theorems.*

These structures consist of a set X, a family $\mathcal{P}$ of distinguished subsets of X, and a family $\mathcal{Z}$ of multivalued self-mappings of X. Their definition is quite similar and is based on two kinds of axioms: a set of axioms for $\mathcal{P}$ and a smaller set of axioms for $\mathcal{Z}$, but the obtained results were mainly directed to encompass fixed point theorems which are attainable by "order-theoretic" techniques, while Definition 3.1 opens the door for a larger variety of fixed point theorems: metric fixed point theorems, topological fixed point theorems etc.

An extension of the above results of Muenzenberger [42], *and Muenzenberger and Smithson* [43] *to common fixed point structures was given by Nelson* [45]

4. Closure mappings

Definition 4.2. *Let X be a nonempty set. A mapping $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be* a closure mapping *if and only if*

(1) $A \subset \eta(A), \forall A \in \mathcal{P}(X)$;
(2) η *is increasing, i.e.* $A \subset B \Rightarrow \eta(A) \subset \eta(B), \forall A, B \in \mathcal{P}(X)$;
(3) $\eta \circ \eta = \eta$.

The following result is useful in the fixed point structure theory.

Lemma 4.1. *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure mapping and $\{A_i\}_{i\in I}$ be a family od subsets of X such tat $\eta(A_i) = A_i$, $i \in I$. Then*

$$\eta\left(\cap_{i\in I} A_i\right) = \cap_{i\in I} A_i (\text{ i.e. }, A_i \in F_\eta, i \in I \Rightarrow A_\infty := \cap_{i\in I} A_i \in F_\eta. \tag{4.1}$$

Definition 4.3. *A fixed point of a closure mapping η will be called* a closed set *or* η-closed set.

Example 4.11. *Let X be a (real or complex) linear space. Then, the following mappings are closure mappings:*

(1) $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(A)$:=*linear hull of A;*
(2) $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(A)$:=*affine hull of A;*
(3) $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(A)$:=*convex hull of A (co A);*

Example 4.12. *Let X be a topological space. The operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(A) = \bar{A}$ is a closure mapping.*

Example 4.13. *Let X be a linear topological space. The operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(A) := co\bar{A} = \overline{co\, A}$ is a closure mapping.*

Remark 4.3. *In general, the composition of two closure mappings is not a closure mapping.*

The following example is illustrating the case of a cartesian product.

Example 4.14. *Let A_i, $i = \overline{1,m}$ be some nonempty sets and $A := \prod_{i=1}^{m} A_i$ their Cartesian product. Let*

$$\pi_i : A \to A_i,\ \pi_i(a_1, a_2, ..., a_m) = a_i,\ i = \overline{1,m}$$

be the canonical projection on A_i.

Let $B \subset A$ be a subset of A. By the Cartesian hull *of a set B we understand the subset*

$$ca\, B := \pi_1(B) \times \pi_2(B) \times ... \times \pi_m(B).$$

The mapping $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$, $\eta(A) = ca\, A$ is a closure mapping.

Remark 4.4. *For more considerations on closure mappings we refer to Rus* [74], *Rus and Şerban* [79], *Kuratowski* [39], *Krasnoselskii and Zabrejko* [37], *Granas and Dugundji* [32], *Rus* [70].

5. Functionals with the intersection property

Definition 5.4. *Let X be a nonempty set, and $Z \subset P(X)$, $Z \neq \emptyset$. A functional $\theta : Z \to \mathbb{R}_+$ has the* intersection property *if the following implication holds:*

$$A_n \in Z,\ A_{n+1} \subset A_n,\ n \in \mathbb{N} \text{ and } \theta(A_n) \to 0 \text{ as } n \to \infty \Rightarrow$$

$$A_\infty := \cap_{n\in\mathbb{N}} A_n \neq \emptyset, A_\infty \in Z \text{ and } \theta(A_\infty) = 0.$$

Remark 5.5. *Let $\theta : Z \to \mathbb{R}_+$ be a functional with the intersection property and let $Z_1 \subset Z$, $Z_1 \neq \emptyset$. Then the functional $\theta|_{Z_1} : Z_1 \to \mathbb{R}_+$ has the intersection property.*

Example 5.15. *Let (X, d) be a metric space. The* diameter functional *$\delta : P_b(X) \to \mathbb{R}_+$ is defined by*

$$\delta(A) := \sup\{d(x, y) | x, y \in A\}.$$

Some properties of the diameter functional are given by the next lemmas.

Lemma 5.2. *Let (X, d) be a metric space and $\delta : P_b(X) \to \mathbb{R}_+$ be the diameter functional. Then*

(i) $\delta(A) = 0 \Leftrightarrow A = \{a\}$;
(ii) $A, B \in P_b(X)$, $A \subset B \Rightarrow \delta(A) \leq \delta(B)$;
(iii) $\delta(\overline{A}) = \delta(A), \forall A \in P_b(X)$.

Lemma 5.3. *Let (X, d) be a complete metric space and $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n, n \in \mathbb{N}$ be such that $\delta(A_n) \to 0$ as $n \to \infty$. Then*

$$\cap_{n\in\mathbb{N}} A_n = \{a^*\},$$

i.e., $\delta : P_{b,cl}(X) \to \mathbb{R}_+$ has the intersection property.

Lemma 5.4. *Let (X, d) be a linear normed space. Then, $\delta : P_b(X) \to \mathbb{R}_+$ has the following properties:*

(i) $\delta(A_1 + A_2) \leq \delta(A_1) + \delta(A_2), \forall A_1, A_2 \in P_{b,cl}(X)$;
(ii) $\delta(\lambda A) = |\lambda|\delta(A), \forall A \in P_{b,cl}(X), \forall \lambda \in \mathbb{R}$;
(iii) $\delta(co\, A) = \delta(A), \forall A \in P_{b,cl}(X)$.

For more considerations on the functional δ, see Kuratowski [39], Rus [70], Rus et al. [77].

Example 5.16 (The Kuratowski measure of noncompactness, Kuratowski [39]**).** *Let (X,d) be a metric space. The functional $\alpha_K : P_{b,cl}(X) \to \mathbb{R}_+$, defined by*

$$\alpha_K(A) := \inf\{\epsilon > 0 | A = \cup_{i=1}^n A_i, \delta(A_i) < \epsilon, \, n \in \mathbb{N}^*\}$$

is called the Kuratowski measure of compactness.

From the definition of Kuratowski measure of noncompactness we have:

Lemma 5.5. *Let (X,d) be a metric space and $\alpha_K : P_{b,cl}(X) \to \mathbb{R}_+$ be the Kuratowski measure of noncompactness. Then*

(i) $A, B \in P_b(X)$, $A \subset B \Rightarrow \alpha_K(A) \leq \alpha_K(B)$;
(ii) $\alpha_K(A \cup B) = \max\{\alpha_K(A), \alpha_K(B)\}, \; A, b \in P_B(X)$;
(iii) $\alpha_K(\overline{a}) = \alpha_K(A), \forall A \in P_b(X)$.

Lemma 5.6. *Let (X,d) be a complete metric space and $\alpha_K : P_{b,cl}(X) \to \mathbb{R}_+$ be the Kuratowski measure of noncompactness. Then we have the following*

(i) $A \in P_b(X)$, $\alpha_K(A) = 0 \Rightarrow \overline{A}$ *is compact;*
(ii)

$$A_n \in P_{b,cl}(X), A_{n+1} \subset A_n, \, n \in \mathbb{N} \text{ and } \alpha_K(A_n) \to 0 \text{ as } n \to \infty \Rightarrow$$

$$A_\infty := \cap_{n \in \mathbb{N}} A_n \neq \emptyset, A_\infty \in P_b(X) \text{ and } \alpha_K(A_\infty) = 0 \text{ and } A_\infty \in P_{cp}(X),$$

i.e., $\alpha_K|_{P_{b,cl}(X)}$ *is a functional with intersection property.*

Lemma 5.7. *Let (X,d) be a linear normed space and α_K be the Kuratowski measure of noncompactness on X. Then*

(i) $\alpha_K(A+B) \leq \alpha_K(A) + \alpha_K(B), \forall A, B \in P_b(X)$;
(ii) $\alpha_K(\lambda A) = |\lambda| \alpha_K(A), \forall A \in P_b(X), \forall \lambda \in \mathbb{R}$;
(iii) $\alpha_K(co\, A) = \delta(A), \forall A \in P_b(X)$.

For the above properties of Kuratowski measure of noncompactness and for other related considerations, see Kuratowski [38], [39], Banaś and Goebel [9], Granas and Dugundji [32], Rus [70], Rus et al. [77], Appell et al. [3], Appell [4], Deimling [25], Nussbaum [47], Turinici [90], Banaś [8], Ariza-Ruiz and García-Falset [5],...

Example 5.17 (The Hausdorff measure of noncompactness, Goldenstein, Gohberg and Markus [31]**).** *Let S and M be subsets of the metric space (X,d) and $\epsilon > 0$. S is called an ϵ*-net *of M in X if, for every $x \in M$, there exists $s \in S$ such that $d(x,s) < \epsilon$. If the set S is finite, then the ϵ-net of M is called a* finite ϵ-net of M, *and we say that M has a finite ϵ-net in X.*

The functional $\alpha_H : P_b(X) \to \mathbb{R}_+$, defined by

$$\alpha_H(A) := \inf\{\epsilon > 0 | A \text{ has a finite } \epsilon\text{-net}\}$$

is called the Hausdorff measure of noncompactness.

Equivalently, the Hausdorff measure of noncompactness α_H can be defined by

$$\alpha_H(A) := \inf\Big\{\epsilon > 0 | \exists x_i \in X, 0 < r_i \leq \epsilon, \, i = \overline{1,m}, m \in \mathbb{N}^* \\ \text{such that } A \subset \cup_{i=1}^m \overline{B}(x_i, r_i)\Big\}.$$

Remark 5.6. *The Hausdorff measure of noncompactness, α_H, also called* set-measure of noncompactness, *possesses all properties of α_K in Lemmas 5.2 and 5.3. Therefore, $\alpha_H : P_b(X) \to \mathbb{R}_+$ is a functional with intersection property.*

Example 5.18 (The De Balsi measure of weak noncompactness, De Blasi [24]**).** *Let X be a Banach space and $A \subset X$ a subset of X. We denote:*

$\overline{A}^w$*:= the weak closure of A;*

$P_{wcl}(X) := \{A \subset X | A$ *is weakly closed* $\}$;

$P_{wcp}(X) := \{A \subset X | A$ *is weakly compact* $\}$.

The functional $\omega_D : P_b(X) \to \mathbb{R}_+$, defined by

$$\omega_D(A) := \inf \left\{\epsilon > 0 | \exists C \in P_{wcp}(X) \text{ such that } A \subset C + \epsilon \overline{B}(0,1)\right\}$$

is called the De Blasi measure of weak noncompactness*.*

By De Blasi intersection Lemma, see De Blasi [24] *and Rus* [70]*, De Blasi measure of noncompactness, $\omega_D|_{P_{wcl,b}}$, has the intersection property.*

Example 5.19 (The Eisenfeld-Lalshmikantham measure of nonconvexity, Eisenfeld and Lakshmikantham [28]**).** *Let (X, d) be a metric space. A continuous function $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if, for all $x, y \in X$ and any $\lambda \in [0,1]$,*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y), \text{ for any } u \in X. \tag{5.2}$$

A metric space (X, d) endowed with a convex structure W is called a Takahashi convex metric space (Takahashi [89]*)and is usually denoted by (X, d, W).*

By definition, a subset $A \subset X$ is convex *if $x, y \in A$ implies $W(x, y; \lambda) \in A$, $\forall \lambda \in [0,1]$.*

A convex metric space is said to have property (c) *if every bounded decreasing net of nonempty, closed and closed subsets has a nonempty intersection.*

By definition, the Eisenfeld-Lalshmikantham measure of nonconvexity on X is the functional $\beta_{EL} : P_b(X) \to \mathbb{R}_+$, defined for all $A \in P_b(X)$ by $\beta_{EL}(A) = H_d(A, co\, A)$, where H_d is the Pompeiu-Hausdorff metric.

By Eisenfeld-Lalshmikantham's intersection Lemma, the functional $\beta_{EL}|_{P_{b,cl}(X)}$ has the intersection property.

For more considerations on measures of nonconvexity, see Eisenfeld and Lakshmikantham [28]*, Rus* [60], [64]*, Takahashi* [89].

6. Compatible pair with respect to a fixed point structure

Let X be a nonempty set, $U \subset P(X)$, $U \neq \emptyset$ and $\theta : U \to \mathbb{R}_+$ a functional. We denote the *zero set of θ* by

$$Z_\theta := \{A \in U | \theta(U) = 0\}.$$

Definition 6.5. *Let $(X, S(X), M)$ be a fixed point structure on X, $Z \subset P(X)$, $Z \neq \emptyset$, $\theta : Z \to \mathbb{R}_+$ and $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$. The pair (θ, η) is said to be* a compatible pair with $(X, S(X), M)$ *if and only if:*

(1) $S(X) \subset \eta(Z) \subset Z$;
(2) η *is a closure mapping;*
(3) $\theta(\eta(A)) = \theta(A), \forall A \in Z$;
(4) $F_\eta \cap Z_\theta \subset S(X)$, *i.e.,* $\theta(A) = 0, \eta(A) = A \Rightarrow A \in S(X)$.

Example 6.20. *Let (X, d) be a metric space and $(X, S(X), M)$ be the trivial fixed point structure on X. Let $\theta := \delta$ and $\eta(A) = \overline{A}$.*

Then (θ, η) is a compatible pair with $(X, S(X), M)$.

Example 6.21. *Let (X, d) be a metric space and $(X, S(X), M)$ be the fixed point structure of contractive mappings on X. If we take $\theta := \alpha_K$ and $\eta(A) = \overline{A}$, then (θ, η) is a compatible pair with $(X, S(X), M)$.*

Example 6.22. *Let X be a Banach space and $(X, S(X), M)$ be the first fixed point structure of Schauder on X. If we take $\theta := \alpha_K$ and $\eta(A) = \overline{co}\, A$, then (θ, η) is a compatible pair with $(X, S(X), M)$.*

For more informations on compatible pairs with a fixed point structure see Rus [63], [65], [70], Berinde et al. [18],...

7. The problem of the invariant fixed point spaces

There are some fixed point problems formulated in terms of the fixed point structure (Rus [70], [73], [74], Rus and Şerban [79], Horvat-Marc and Berinde [33],...)
One of them in given by the following

Problem 7.1 Let $(X, S(X), M)$ be a fixed point structure on X and $A \in P(X)$. For which mappings $f : A \to A$ there exists $B \in I(f) \cap P(A)$ such that $B \in S(X)$ and $f|_B \in M(B)$?

In the case of some fixed point structure there exist specific results. Some examples are given in the following.

Example 7.23. *Let $X := C[a,b]$ endowed with a suitable Bielecki norm $\|\cdot\|_\tau$ and let $(C[a,b], P_{cl}(C[a,b]), M)$ be the fixed point structure of contractions on $(C[a,b], \|\cdot\|_\tau)$. Consider the mapping $f : C([a,b]) \to C[a,b]$ defined by*

$$f(x)(t) := x(a) + \int_a^t K(t,s,x(s))ds,\ t \in [a,b],$$

where $K \in C\left([a,b] \times [a,b] \times \mathbb{R}\right)$ satisfies

$$|K(t,s,u) - K(t,s,v)| \le L_K|u-v|,\ \forall s,t \in [a,b], \forall u,v \in \mathbb{R},$$

for some constant $L_K > 0$.

The problem is to find conditions that ensure the existence of $A \in P_{cl}(C[a,b]) \cap I(f)$ such that $f|_A : A \to A$ is a contraction.

First of all, we remark that

$$|f(x)(t) - f(y)(t)| \le |x(a) - y(a)| + \int_a^t |K(t,s,x(s)) - K(t,s,y(s))|ds$$

$$\le |x(a) - y(a)| + L_K \int_a^t |x(s) - y(s)|ds \le |x(a) - y(a)| +$$

$$+ L_K \int_a^t |x(s) - y(s)| e^{-\tau(s-a)} e^{\tau(s-a)} ds$$

$$\le |x(a) - y(a)| + L_K \|x - y\|_\tau \int_a^t e^{\tau(s-a)} ds$$

$$\le |x(a) - y(a)| + \frac{L_K}{\tau} \|x - y\|_\tau e^{-\tau(t-a)}, \forall x, y \in C[a,b]. \tag{7.3}$$

From (7.3) *we have*

$$|f(x)(t) - f(y)(t)| e^{-\tau(t-a)} \le |x(a) - y(a)| e^{-\tau(t-a)} + \frac{L_K}{\tau} \|x - y\|_\tau,$$

i.e.,

$$\|f(x) - f(y)\|_\tau \le \frac{L_K}{\tau} \|x - y\|_\tau,\ \forall x, y \in C[a,b] \text{ with } x(a) = y(a).$$

For $\lambda \in \mathbb{R}$, let us denote

$$X_\lambda := \{x \in C[a,b] | x(a) = \lambda\}.$$

From the above results it is clear that

- $C[a, b] = \cup_{x \in \mathbb{R}} X_\lambda$
- $f(X_\lambda) \subset X_\lambda$
- *if we choose* $\tau > 0$*, then the mapping* $f|_{X_\lambda}$ *is a contraction.*

Therefore, in this case, $C[a, b] = \cup_{x \in \mathbb{R}} X_\lambda$ *is a partition with* X_λ f*-invariant fixed point spaces.*

For other specific examples related to f-invariant fixed point spaces, see Rus [73], [74], Rus et al. [77], Rus [69], Petruşel et al. [53],...

The problem is to give an abstract result for Problem 7.1.

8. (θ, φ)-CONTRACTIONS

In this section we present a class of mappings through which we can provide a solution to Problem 7.1.

Definition 8.6. *A function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is called* a comparison function *if it satisfies the following conditions:*

(1_φ) φ *is nondecreasing;*

(2_φ) $\varphi^n(t) \to 0$ *as* $n \to \infty$, $\forall t \in \mathbb{R}_+$*, where* $\varphi^{n+1} := \varphi(\varphi^n)$.

Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$ and $\theta : Z \to \mathbb{R}_+$ be a functional.

Definition 8.7. *A mapping* $f : X \to X$ *is said to be a* strong (θ, φ)-contraction *if and only if:*

(1) φ *is a comparison function;*

(2) $A \in Z \Rightarrow f(A) \in Z$;

(3) $\theta(f(A)) \leq \varphi(\theta(A)), \forall A \in Z$.

Definition 8.8. *A mapping* $f : X \to X$ *is said to be a* (θ, φ)-contraction *if (1) and (2) in Definition 8.7 hold and we have*

$(3)'$ $\theta(f(A)) \leq \varphi(\theta(A)), \forall A \in Z \cap I(f)$.

Remark 8.7. *Let* $0 < l < 1$*. Then* $\varphi(t) = lt$, $t \in \mathbb{R}_+$*, is a comparison function. In this case we simply denote the* (θ, φ)*-contraction as* (θ, l)*-contraction.*

Remark 8.8. *For some examples of* (δ, φ)*-contractions,* (α, φ)*-contractions,* (d_H, φ)*-contractions,* (β_{EL}, φ)*-contractions, and* (ω_D, φ)*-contractions, see Rus* [70]*, Banaś and Goebel* [9]*, Rus* [60]*,* [64]*, Rus and Şerban* [79]*, Appell et al.* [3]*, Nussbaum* [47]*, Turinici* [90]*, Horvat-Marc and Berinde* [33]*, Rus* [69]*, Banaś* [8]*, Asadi et al.* [6]*, Ariza-Ruiz and García-Falset* [5]*,...*

9. FIXED POINT PRINCIPLE OF (θ, φ)-CONTRACTIONS

Now we have all concepts which are necessary to present the first general fixed point principle of the fixed point structure theory.

Theorem 9.2 (**(θ, φ)-contraction principle**, Rus [70])**.** *Let* $(X, S(X), M)$ *be a fixed point structure and* (θ, η) *a compatible pair with* $(X, S(X), M)$*. Let* $A \in \eta(Z)$ *and* $f \in M(A)$*. Suppose that:*

(1) $\theta|_{\eta(Z)}$ *has the intersection property;*

(2) f *is a* (θ, φ)*-contraction.*

Then we have the following:

(i) $I(f) \cap S(X) \neq \emptyset$*, i.e.,* f *has at least an invariant fixed point space;*

(ii) $F_f \neq \emptyset$*;*

(iii) If $F_f \in Z$*, then* $\theta(F_f) = 0$.

Proof. (see Rus [70])

"(i)+(ii)".

$A \in \eta(Z)$ implies $A \in Z$. Assumption (2) implies that $f(A) \in Z$. Now, let

$$A_1 := \eta(f(A)), A_2 := \eta(f(A_1)), \ldots, A_{n+1} := \eta(f(A_n)), ..., n \in \mathbb{N}^*.$$

By the definition of the sequence $\{A_n\}$ we note that

$$A_{n+1} \subset A_n,\ A_n \in F_\eta \text{ and } A_n \in I(f).$$

By assumption (2) it follows:

$$\theta(A_{n+1}) = \theta(\eta(f(A_n))) = \theta(f(A_n)) \leq \varphi(\theta(A_n)) \leq \ldots$$
$$\leq \varphi^{n+1}(\theta(A)) \to 0 \text{ as } n \to \infty.$$

Since $\theta : \eta(Z) \to \mathbb{R}_+$ is a functional with the intersection property, we have that

$$A_\infty := \cap_{n\in\mathbb{N}^*} A_n \neq \emptyset,\ A_\infty \in \eta(Z), A_\infty \in I(f) \text{ and } \theta(A_\infty) = 0.$$

Having in view the fact that (θ, η) is compatible with $(X, S(X), M)$, we have $A_\infty \in I(f) \cap S(X)$ and $f|_{A_\infty} \in m(A_\infty)$.

So, $F_f \neq \emptyset$.

"(iii)"

Let $F_f \in Z$. By the assumption (2) and from the fact that $f(F_f) = F_f$ it follows that

$$\theta(F_f) = \theta(f(F_f)) \leq \varphi(\theta(F_f)).$$

Now, since φ is a comparison function, we have that $\theta(F_f) = 0$. □

Remark 9.9. *In Theorem 9.2, we can replace the assumption "$A \in \eta(Z)$" by "$A \in F_\eta$ and $f(A) \in Z$".*

Remark 9.10. *For the corresponding abstract version of Theorem 9.2 in the case of condensing mappings, see Rus* [70].

10. Corollaries and new possible results

If in Theorem 9.2 we take (X, d) to be a bounded and complete metric space, $(X, S(X), M)$ be the trivial fixed point structure on X, $Z := P(X)$, $\theta := \delta$, $\eta := \overline{(\cdot)}$ and $A := X$, the we have

Theorem 10.3. *Let (X, d) be bounded and complete metric space and $f : X \to X$ a (δ, φ)-contraction.*

Then $F_f = \{x^*\}$.

Additionally, we have the following:

(i) $f^n(x) \to x^*$ as $n \to \infty$, i.e., f is a Picard mapping;

(ii) $\cap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$, i.e., f is a Janos mapping.

Proof. "(i)"

If we denote

$$A_n := \{f^n(x), f^{n+1}(x), \ldots\} \cup \{x^*\}$$

then we observe that

$$A_n \in P_{b,cl}(X) \cap I(f) \text{ and } \delta(A_{n+1} \leq \varphi^n(A_1)) \to 0 \text{ as } n \to \infty,$$

i.e.,

$$\delta(A_\infty) := \delta\left(\cap_{n\in\mathbb{N}} A_n\right) = 0 \Rightarrow A_\infty = \{x^*\}$$

that is, $f^n(x) \to x^*$ as $n \to \infty$, and hence f is a Picard mapping.

"(ii)"

Let

$$X_1 := f(X),\ X_2 := f(X_1), \dots, X_{n+1} := f(X_n), \dots$$

We observe that

- $\delta(X_{n+1}) \leq \varphi(X_1) \to 0$ as $n \to \infty$;
- $\cap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$, i.e., f is a Janos mapping.

$\square$

Remark 10.11. *From Theorem 10.3 we recover almost all fixed point principles for generalized contractions (Berinde* [11]*, Berinde et al.* [18]*, Rus* [70]*, Rus et al.* [77]*, Rus* [69]*,...).*

If in Theorem 9.2 we consider X a Banach space, $(X, S(X), M)$ be the first fixed point structure of Schauder, $Z := P_b(X)$, $\theta := \alpha_K$, $\eta := \overline{co}$, then we have

Theorem 10.4. *Let X be a Banach space, α_K the Kuratowski measure of noncompactness on X, $A \in P_{b,cl,cv}(X)$ and $f : A \to A$ a mapping. We suppose that:*

(1) $f \in C(A, A)$;
(2) f is an (α_K, φ)-contraction.

Then we have that

(i) $F_f \neq \emptyset$;
(ii) F_f is a compact subset of A.

Remark 10.12. *Theorem 10.4 is a generalization of Darbo's fixed point theorem. In the original Darbo's fixed point theorem (Darbo* [23]*), f is supposed to be a strong (α_K, φ)-contraction, while here we assume that f is an (α_K, φ)-contraction.*

Remark 10.13. *For other corollaries of Theorem 9.2 (.e.g., $\theta := \alpha_H$, $\theta := \omega_D$, $\theta := \beta_{EL}$,...) see Rus* [70] *and the references therein.*

Problem 10.1. To look on Theorem 9.2 as a source for new fixed point theorems.

For other open problems in the fixed point theory in terms of fixed point structures (and also an extensive bibliography on the subject), see Rus [72].

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