

# A unified iterative scheme for approximating fixed points of Berinde nonexpansive mappings in uniformly convex Banach spaces

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**ABSTRACT.** We introduce a unified iterative scheme that strictly includes several classical fixed point algorithms, including the Mann, Krasnoselskii, Picard–Mann, Picard–Krasnoselskii, and related multi-step iterations. Under mild control conditions in uniformly convex Banach spaces, we prove that every sequence generated by this scheme is an approximate fixed point sequence. Building on this foundational property, we establish strong convergence theorems for Berinde nonexpansive mappings satisfying condition  $(\bar{T})$  and for contraction mappings without any additional assumptions. Finally, we present applications in Hilbert spaces via metric projections, yielding strong convergence to fixed points located within the closed convex set of attractive points.

## 1. INTRODUCTION

Fixed point iteration is a central methodology for solving nonlinear equations and inequalities in metric and Banach space settings. Canonical procedures such as the Picard, Mann, and Krasnoselskii iterations, and their hybrid variants, have been widely studied to establish convergence under various contractive or nonexpansive-type assumptions; see [4, 14, 17]. Beyond nonexpansiveness, Berinde’s weak contraction framework provides a broader class of mappings, with special emphasis on the  $(1, L)$ -case (often called Berinde nonexpansive mappings), where nonexpansivity is relaxed through a controlled defect term; see [1, 2, 3].

Another concept—attractive points—has proved useful in relating approximation behavior to geometric properties of mappings and spaces. Initially investigated in Hilbert spaces; see [22], the theory of attractive points has been extended to Banach and CAT(0) settings; see [10, 15, 16, 28]. In uniformly convex Banach spaces, attractive points provide a powerful lens for establishing monotonicity and for deriving approximate fixed point sequences, which often serve as a critical bridge to strong convergence; see [29].

Motivated by these developments, we propose a unified iterative scheme that captures a broad family of classical processes within a single convex-closure-and-iteration framework. The key structural feature of our scheme is that, at each iterate, we permit updates to lie in the iterated convex hull  $S^\infty(\cdot)$  of forward images, thereby subsuming a wide range of known methods. Our first main result shows that, under mild control on the step parameters and assuming the existence of an attractive point, every sequence generated by the scheme is an approximate fixed point sequence of a mapping  $T$  (i.e., the residuals  $\|x_n - Tx_n\|$  converge to 0). We then leverage this general property to prove strong convergence for (i) Berinde nonexpansive mappings satisfying condition  $(\bar{T})$ , and (ii) contraction mappings, where the latter requires no additional conditions beyond the control on the step sequence. In Hilbert spaces, the convexity and closedness of the set of attractive

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points allow us to combine our scheme with metric projections to obtain strong convergence to fixed points contained in the intersection of the feasible set with the attractive point set. This unifies the treatment of several classical iterations under a single abstract scheme and highlights the role of attractive points in organizing convergence arguments. Concretely, our contributions are:

- We define a unified iterative scheme that strictly includes several classical fixed point methods (Mann, Krasnoselskii, Picard–Mann, Picard–Krasnoselskii, and multi-step variants).
- We prove that, under mild control conditions and assuming the existence of an attractive point in uniformly convex Banach spaces, every sequence generated by the scheme is an approximate fixed point sequence.
- We establish strong convergence theorems for Berinde nonexpansive mappings satisfying condition  $(\bar{I})$ .
- We derive strong convergence for contraction mappings with only a lower bound on the control parameters.
- In Hilbert spaces, we combine the scheme with metric projections to obtain strong convergence to fixed points within the closed convex set of attractive points.

The paper is organized as follows. Section 2 recalls preliminaries on uniformly convex spaces, attractive points, and metric projections, and fixes notation. Section 3 introduces the unified iterative scheme, proves the approximate fixed point theorem, and then develops strong convergence results for Berinde nonexpansive and contraction mappings. Section 4 presents Hilbert-space applications via metric projections. Section 5 concludes.

## 2. PRELIMINARIES

In this section, we present the definitions and fundamental lemmas that will be used throughout the paper. Throughout this work, we use  $\mathbb{R}$  to denote the set of real numbers, and  $\mathbb{N}$  the set of positive integers. Let  $C$  be a nonempty closed convex subset of a Banach space  $X$ . Let  $T : C \rightarrow C$  be a mapping, we denote  $F(T)$  by the set of all fixed points of  $T$ , i.e.,  $F(T) = \{x \in C : x = Tx\}$ . A mapping  $T : C \rightarrow C$  is said to be nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . Similarly,  $T$  is said to be a contraction if there exists a constant  $c \in (0, 1)$  such that  $\|Tx - Ty\| \leq c\|x - y\|$  for all  $x, y \in C$ . Berinde [3] introduced weak contractions: A mapping  $T$  on a metric space  $(X, d)$  is called a  $(\delta, L)$ -almost contraction if  $\delta \in (0, 1)$ ,  $L \geq 0$ , and for all  $x, y \in X$ ,

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

In Banach spaces, the case  $\delta = 1$  is known as a Berinde nonexpansive mapping (or  $(1, L)$ -almost contraction): for all  $x, y \in X$ ,

$$\|Tx - Ty\| \leq \|x - y\| + L\|y - Tx\|,$$

see [6, 13]. Every nonexpansive mapping is a  $(1, 0)$ -almost contraction.

A mapping  $T : C \rightarrow X$  is called quasi-nonexpansive if  $\|Tx - z\| \leq \|x - z\|$ , for all  $x \in C$  and for all  $z \in F(T)$ . Hence, every nonexpansive mapping that has a fixed point is a quasi-nonexpansive mapping. Additionally, the example of a Berinde nonexpansive mapping that is neither quasi-nonexpansive nor nonexpansive is as follows:

**Example 2.1.** [6] Let  $X = \mathbb{R}$ ,  $C = [0, 1]$  and a mapping  $T : C \rightarrow C$  defined by

$$Tx = \begin{cases} x^2, & \text{if } x \in [0, \frac{1}{2}), \\ 1, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then the mapping  $T$  is a  $(1, 4)$ -almost contraction, but it is not nonexpansive. Moreover,  $F(T) = \{0, 1\}$ , and  $T$  is not quasi-nonexpansive. Indeed, for the pair  $(\frac{1}{3}, 1) \in C \times F(T)$ , we observe that

$$\left| T\left(\frac{1}{3}\right) - 1 \right| > \left| \frac{1}{3} - 1 \right|.$$

Recall that a sequence  $\{x_n\}$  in  $C$  is called an *approximate fixed point sequence* for a mapping  $T$  if

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

A Banach space  $X$  is called *uniformly convex* if for every  $\varepsilon$ , with  $0 < \varepsilon \leq 2$ , there is  $\delta(\varepsilon) > 0$  such that

$$\|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \text{ imply } \frac{\|x + y\|}{2} < 1 - \delta(\varepsilon).$$

Note that uniform convexity implies strict convexity and reflexivity; Hilbert spaces are uniformly convex (see [7]).

Let  $\phi : C \rightarrow \mathbb{R}$  be a convex function and suppose  $x_1, x_2, \dots, x_m \in C$ , and  $\gamma_1, \gamma_2, \dots, \gamma_m \in [0, 1]$  with  $\sum_{j=1}^m \gamma_j = 1$ , where  $m \in \mathbb{N}$ . Then the following inequality holds:

$$\phi\left(\sum_{j=1}^m \gamma_j x_j\right) \leq \sum_{j=1}^m \gamma_j \phi(x_j),$$

which is the standard form of Jensen’s inequality. In particular, if  $\phi(x) = \|x\|$ , then Jensen’s inequality implies

$$\left\| \sum_{j=1}^m \gamma_j x_j \right\| \leq \sum_{j=1}^m \gamma_j \|x_j\|.$$

The following lemma is required to prove the main theorem:

**Lemma 2.1.** [5] *Let  $X$  be a uniformly convex Banach space and  $B_r = \{x \in X : \|x\| \leq r\}$ ,  $r > 0$ . Then there exists a continuous, strictly increasing, and convex function  $g : [0, \infty) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that*

$$\|\lambda x + \beta y + \alpha z\|^2 \leq \lambda \|x\|^2 + \beta \|y\|^2 + \alpha \|z\|^2 - \lambda \beta g(\|x - y\|),$$

for all  $x, y, z \in B_r$  and for all  $\lambda, \beta, \alpha \in [0, 1]$  with  $\lambda + \beta + \alpha = 1$ .

Another concept closely related to fixed points is that of attractive points, first introduced in Hilbert spaces by Takahashi and Takeuchi [22]. Let  $H$  be a Hilbert space, and let  $T : C \rightarrow H$  be a mapping, where  $C$  is a nonempty subset of  $H$ . The set of all attractive points of  $T$  is denoted by  $A(T)$ , and is defined by

$$A(T) = \{z \in H : \|z - Ty\| \leq \|z - y\|, \forall y \in C\}.$$

Intuitively, each  $z \in A(T)$  attracts points under  $T$  in the sense that applying  $T$  to any  $y \in C$  does not increase its distance from  $z$ ; see [9, 12, 25, 27] and references therein for more results.

The significance of attractive points lies in their connection to fixed points. Takahashi and Takeuchi [22] proved that the attractive points can guarantee the existence of fixed points under mild conditions as follows:

**Lemma 2.2.** *Let  $C$  be a nonempty closed convex subset of  $H$ , and let  $T$  be a mapping from  $C$  into itself. If  $A(T) \neq \emptyset$ , then  $F(T) \neq \emptyset$ . In particular, if  $z \in A(T)$ , then  $P_C z \in F(T)$ , where  $P_C$  is the metric projection from  $H$  onto  $C$ .*

Moreover, the important structure of  $A(T)$  is also proved by the authors [22].

**Lemma 2.3.**  $A(T)$  is a closed convex subset of  $H$ .

In the case of quasi-nonexpansive mappings, the relationship between attractive points and fixed points becomes more precise:

**Lemma 2.4.** [24] Let  $C$  be a nonempty subset of a real Hilbert space  $H$ . If  $T : C \rightarrow H$  be a quasi-nonexpansive mapping from  $C$  into  $H$ , then  $A(T) \cap C = F(T)$ .

Lemma 2.3 and Lemma 2.4 are also true in Banach spaces, see Lemma 9 and Lemma 10 of [29]. More generally, we have the following containment relationship:

**Remark 2.1.**  $A(T) \cap C \subset F(T)$ .

*Proof.* Let  $z \in A(T) \cap C$ . Then  $z \in A(T)$ , i.e.,

$$\|Ty - z\| \leq \|y - z\|, \text{ for all } y \in C.$$

In particular, by choosing  $y = z \in C$ , we get

$$\|Tz - z\| \leq \|z - z\| = 0.$$

It follows that  $z \in F(T)$ . □

However, there is no containment relationship between  $F(T)$  and  $A(T)$  in general. The following example shows that  $F(T) \not\subseteq A(T)$  and  $A(T) \not\subseteq F(T)$ .

**Example 2.2.** [26] Let  $H = \mathbb{R}$  and  $C = [0, 1]$ . Define a mapping  $T : C \rightarrow C$  by

$$Tx = x^2, \text{ for all } x \in C.$$

Then  $F(T) = \{0, 1\}$ , and  $A(T) = (-\infty, 0]$ .

For a given nonempty closed convex subset  $C$  in a Hilbert space  $H$ , the metric projection mapping  $P$  from  $H$  onto  $C$  is defined as follows: for any point  $u \in H$ ,  $P_C u$  is the unique point in  $C$  that minimizes the distance to  $u$ . In mathematical terms, it satisfies the property:

$$\|x - P_C x\| = d(x, C) = \inf\{\|x - p\| : p \in C\}.$$

It is known that the metric projection mapping  $P$  is nonexpansive, and satisfies

$$\langle x - P_C x, P_C x - p \rangle \geq 0,$$

for all  $x \in H$  and  $p \in C$ . For more details, see [20].

To prove our main result in this paper we need the following useful lemma:

**Lemma 2.5.** [23] Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $P$  be a metric projection of  $H$  onto  $C$ , and let  $\{x_n\}$  be a sequence in  $H$ . If  $\|x_{n+1} - z\| \leq \|x_n - z\|$  for all  $n \in \mathbb{N}$  and for all  $z \in C$ , then  $\{P x_n\}$  converges strongly to some  $y \in C$ .

### 3. UNIFIED ITERATIVE SCHEME AND MAIN RESULTS

**3.1. The unified iterative scheme.** Let  $X$  be a uniformly convex Banach space, and let  $T : C \rightarrow C$  be a mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$ . For  $D \subset X$ , define

$$S(D) := \text{co}\{T^k y : y \in D, k \in \mathbb{N} \cup \{0\}\},$$

where “co” means the convex hull operator. Observe that

$$(1) \quad D \subset S(D) \subset S^2(D) \subset S^3(D) \subset \dots$$

Indeed, let  $x \in D$ . Then  $x = T^0x \in S(D)$ , which implies that  $D \subset S(D)$ . Thus,  $S(D) \subset S(S(D)) = S^2(D)$ . Repeating this process shows that (1) holds. We also define

$$S^\infty(D) = \bigcup_{i=1}^\infty S^i(D).$$

For a singleton  $D = \{x\}$ , write  $S(x)$ ,  $S^i(x)$ , and  $S^\infty(x)$ . Using the notations, we can see that if  $x_1 \in C$  and  $z_n = (1 - \gamma_n)x_n + \gamma_nTx_n$ , then  $y_n = (1 - \beta_n)z_n + \beta_nTz_n \in S(z_n)$ , and  $x_{n+1} = (1 - \alpha_n)Tz_n + \alpha_nTy_n \in S(S(z_n)) = S^2(z_n)$ , for all  $n \in \mathbb{N}$ .

We can construct a general iterative scheme as follows:

**Definition 3.1** (Unified iterative scheme). *Given  $x_1 \in C$  and a sequence  $\{\gamma_n\}$  in  $[0, 1]$ ,*

$$(2) \quad \begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ x_{n+1} \in S^\infty(z_n), \text{ for all } n \in \mathbb{N}. \end{cases}$$

**Remark 3.2** (Special cases). *The iterative process (2) includes as special cases several well-known iterative methods, such as the Mann iterative process, the Krasnoselskii iterative process, the Picard–Mann hybrid iterative process [11], the Picard–Krasnoselskii hybrid iterative process [18], and the iterative process (3) defined by [6], as follows:*

(i) *Mann iterative process:*

$$x_{n+1} = (1 - \gamma_n)x_n + \gamma_nTx_n, \text{ for all } n \in \mathbb{N},$$

where  $\{\gamma_n\}$  is a sequence in  $[0, 1]$ .

(ii) *Krasnoselskii iterative process: as Mann with constant  $\gamma_n = \gamma$ .*

(iii) *Picard–Mann hybrid iterative process:*

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ x_{n+1} = Tz_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\gamma_n\}$  is a sequence in  $[0, 1]$ .

(iv) *Picard–Krasnoselskii iterative process: as Picard–Mann with constant  $\gamma_n = \gamma$ .*

(v) *The three-step iterative process:*

$$(3) \quad \begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ y_n = (1 - \beta_n)z_n + \beta_nTz_n, \\ x_{n+1} = (1 - \alpha_n)Tz_n + \alpha_nTy_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  are sequences in  $[0, 1]$ .

To establish an approximate fixed point result under the iterative scheme (2), the following property is required:

**Proposition 3.1.** *If  $w \in S^\infty(x)$  and  $u \in A(T)$ , then  $\|w - u\| \leq \|x - u\|$ .*

*Proof.* Let  $P(n)$  be the statement : If  $w \in S^n(x)$  and  $u \in A(T)$ , then  $\|w - u\| \leq \|x - u\|$ .

Base step : For  $n = 1$ , if  $w \in S(x)$  and  $u \in A(T)$ , then  $w = \sum_{j=1}^m \gamma_j T^{k_j} x$ , where  $\sum_{j=1}^m \gamma_j = 1$ ,  $\gamma_j \in [0, 1]$ ,  $m \in \mathbb{N}$ , and  $k_j \in \mathbb{N} \cup \{0\}$ . By Jensen’s inequality,  $u \in A(T)$ , and  $w \in S(x)$ , we

get

$$\begin{aligned}
 \|w - u\| &= \left\| \left( \sum_{j=1}^m \gamma_j T^{k_j} x \right) - u \right\| \\
 &= \left\| \sum_{j=1}^m \gamma_j (T^{k_j} x - u) \right\| \\
 &\leq \sum_{j=1}^m \gamma_j \|T^{k_j} x - u\| \\
 &\leq \sum_{j=1}^m (\gamma_j \|x - u\|) \\
 &= \|x - u\| \sum_{j=1}^m \gamma_j \\
 &= \|x - u\|.
 \end{aligned}$$

Thus,  $P(1)$  holds.

**Inductive step :** Assume that  $P(k)$  holds for some  $k \in \mathbb{N}$ . We will show that  $P(k + 1)$  holds. Let  $u \in A(T)$  and  $w \in S^{k+1}(x)$ . Then  $w = \sum_{j=1}^m \gamma_j T^{l_j} w_j$ , where  $w_j \in S^k(x)$ ,

$\sum_{j=1}^m \gamma_j = 1$ ,  $\gamma_j \in [0, 1]$ ,  $m \in \mathbb{N}$ , and  $l_j \in \mathbb{N} \cup \{0\}$ . Since  $w_j \in S^k(x)$  and  $P(k)$  holds, we have  $\|w_j - u\| \leq \|x - u\|$ . From Jensen's inequality,  $u \in A(T)$ , and  $w_j \in S^k(x)$ , we obtain

$$\begin{aligned}
 \|w - u\| &= \left\| \left( \sum_{j=1}^m \gamma_j T^{l_j} w_j \right) - u \right\| \\
 &= \left\| \sum_{j=1}^m \gamma_j (T^{l_j} w_j - u) \right\| \\
 &\leq \sum_{j=1}^m \gamma_j \|T^{l_j} w_j - u\| \\
 &\leq \sum_{j=1}^m \gamma_j \|w_j - u\| \\
 &\leq \sum_{j=1}^m \gamma_j \|x - u\| \\
 &= \|x - u\| \sum_{j=1}^m \gamma_j \\
 &= \|x - u\|.
 \end{aligned}$$

Therefore,  $P(k + 1)$  holds. By the principle of mathematical induction,  $P(n)$  holds for all  $n \in \mathbb{N}$ .

Observe that, if  $w \in S^\infty(x)$  and  $u \in A(T)$ , then there exists some  $i \in \mathbb{N}$  such that  $w \in S^i(x)$ . From  $u \in A(T)$ ,  $w \in S^i(x)$  and  $P(i)$  holds implies that  $\|w - u\| \leq \|x - u\|$ , and hence the proof is completed.  $\square$

For proving our main theorem in this paper, we also need the following lemma:

**Lemma 3.6.** *Let  $T : C \rightarrow C$  be a mapping defined on a nonempty subset  $C$  of a uniformly convex Banach space  $X$ . If the sequence  $\{x_n\}$  satisfies (2), i.e., let  $x_1 \in C$ , and*

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ x_{n+1} \in S^\infty(z_n), \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\gamma_n\}$  is a sequence in  $[0, 1]$ . Then for all  $u \in A(T)$ ,

- (i)  $\|x_{n+1} - u\| \leq \|z_n - u\| \leq \|x_n - u\|$ , for all  $n \in \mathbb{N}$ ;
- (ii)  $\lim_{n \rightarrow \infty} \|x_n - u\|$  exists.

*Proof.* From  $u \in A(T)$ ,  $z_n \in S(x_n)$ , and Proposition 3.1, we get

$$(4) \quad \|z_n - u\| \leq \|x_n - u\|, \text{ for all } n \in \mathbb{N}.$$

It follows from  $u \in A(T)$ ,  $x_{n+1} \in S^\infty(z_n)$ , and Proposition 3.1, we get

$$(5) \quad \|x_{n+1} - u\| \leq \|z_n - u\|, \text{ for all } n \in \mathbb{N}.$$

Combining (4) and (5), we obtain

$$(6) \quad \|x_{n+1} - u\| \leq \|z_n - u\| \leq \|x_n - u\|, \text{ for all } n \in \mathbb{N}.$$

This means that  $\{\|x_n - u\|\}$  is nonincreasing and bounded below. Therefore,  $\lim_{n \rightarrow \infty} \|x_n - u\|$  exists. □

**3.2. Approximate fixed point sequences under the unified scheme.** The following lemma is our first main result: every sequence generated by the unified scheme (2) is an approximate fixed point sequence under mild conditions, provided attractive points exist.

**Lemma 3.7.** *Let  $T : C \rightarrow C$  be a mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$  with  $A(T) \neq \emptyset$ . Suppose that the sequence  $\{x_n\}$  satisfies (2) such that  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ . Then  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ .*

*Proof.* Let  $u \in A(T)$ . From Lemma 3.6, the sequence  $\{\|x_n - u\|\}$  is nonincreasing and bounded below. Hence,

$$\|x_n - u\| \leq \|x_1 - u\|, \text{ for all } n \in \mathbb{N}.$$

Since  $u \in A(T)$ , we get

$$\|Tx_n - u\| \leq \|x_n - u\| \leq \|x_1 - u\|, \text{ for all } n \in \mathbb{N}.$$

Thus,  $\{x_n - u\}, \{Tx_n - u\} \subset B_{\|x_1 - u\|}$ .

By Lemma 2.1, there exists a continuous, strictly increasing, and convex function  $g : [0, \infty) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that

$$\begin{aligned} \|z_n - u\|^2 &= \|(1 - \gamma_n)(x_n - u) + \gamma_n(Tx_n - u)\|^2 \\ &\leq (1 - \gamma_n)\|x_n - u\|^2 + \gamma_n\|Tx_n - u\|^2 - (1 - \gamma_n)\gamma_n g(\|x_n - Tx_n\|) \\ &\leq (1 - \gamma_n)\|x_n - u\|^2 + \gamma_n\|x_n - u\|^2 - (1 - \gamma_n)\gamma_n g(\|x_n - Tx_n\|) \\ &= \|x_n - u\|^2 - (1 - \gamma_n)\gamma_n g(\|x_n - Tx_n\|), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By (5) of Lemma 3.6, we obtain

$$\|x_{n+1} - u\|^2 \leq \|z_n - u\|^2 \leq \|x_n - u\|^2 - (1 - \gamma_n)\gamma_n g(\|x_n - Tx_n\|), \text{ for all } n \in \mathbb{N}.$$

Hence,

$$(7) \quad (1 - \gamma_n)\gamma_n g(\|x_n - Tx_n\|) \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2, \text{ for all } n \in \mathbb{N}.$$

By assumption on the control sequence  $\{\gamma_n\}$ , there exist  $n_0 \in \mathbb{N}$ , and constants  $\eta, \kappa \in (0, 1)$  such that  $0 < \eta \leq \gamma_n \leq \kappa < 1$ , for all  $n \geq n_0$ . It follows from (7) that

$$0 \leq (1 - \kappa)\eta g(\|x_n - Tx_n\|) \leq (1 - \gamma_n)\gamma_n g(\|x_n - Tx_n\|) \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Since  $\lim_{n \rightarrow \infty} \|x_n - u\|$  exists, we get  $\lim_{n \rightarrow \infty} (1 - \kappa)\eta g(\|x_n - Tx_n\|) = 0$ , and hence

$$\lim_{n \rightarrow \infty} g(\|x_n - Tx_n\|) = 0.$$

Since  $g$  is continuous, strictly increasing, and  $g(0) = 0$ , we have

$$(8) \quad \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

□

**Remark 3.3.** *Since the Mann iterative process, the Krasnoselskii iterative process, the Picard–Mann hybrid iterative process, the Picard–Krasnoselskii hybrid iterative process, and the iterative process (3) are included in the iterative scheme (2), every sequence  $\{x_n\}$  generated by these iterative processes is an approximate fixed point sequence, if the control conditions  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$  hold.*

The following example shows that any sequence constructed by the following process also satisfies the iterative process (2), and hence it is an approximate fixed point sequence:

**Example 3.3.** *Let  $T : C \rightarrow C$  be a mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$  with  $A(T) \neq \emptyset$ . Suppose that the sequence  $\{x_n\}$  defined by  $x_1 \in C$ , and*

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ y_n = (1 - \beta_n)z_n + \beta_nTz_n, \\ u_n = (1 - \delta_n)y_n + \delta_nTy_n, \\ v_n = (1 - \lambda_n)u_n + \lambda_nTu_n, \\ x_{n+1} = (1 - \alpha_n)v_n + \alpha_nTv_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\gamma_n\}, \{\beta_n\}, \{\delta_n\}, \{\lambda_n\}, \{\alpha_n\} \subset [0, 1]$ , and  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ . Then  $\{x_n\}$  satisfies the unified iterative process (2). Indeed,

$$\begin{aligned} y_n &= (1 - \beta_n)z_n + \beta_nTz_n \in S(z_n), \\ u_n &= (1 - \delta_n)y_n + \delta_nTy_n \in S(S(z_n)) = S^2(z_n), \\ v_n &= (1 - \lambda_n)u_n + \lambda_nTu_n \in S(S^2(z_n)) = S^3(z_n), \\ x_{n+1} &= (1 - \alpha_n)v_n + \alpha_nTv_n \in S(S^3(z_n)) = S^4(z_n), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence,  $\{x_n\}$  satisfies the unified iterative process defined in (2). By Lemma 3.7, we conclude that  $\{x_n\}$  is an approximate fixed point for the mapping  $T$ .

**3.3. Strong convergence for Berinde nonexpansive mappings.** Let  $C$  be a nonempty subset of a Banach space  $X$ . A mapping  $T : C \rightarrow C$  is said to satisfy condition  $(\bar{I})$  if there is a nondecreasing function  $h : [0, \infty) \rightarrow [0, \infty)$  with  $h(0) = 0$  and  $h(r) > 0$  for all  $r > 0$  such that

$$0 \leq h(d(x, A(T))) \leq \|x - Tx\|, \quad \forall x \in C,$$

where  $d(x, A(T)) = \inf\{d(x, y) : y \in A(T)\}$ .

Next, using the existence of an attractive point, we prove the strong convergence theorem of fixed points and attractive points for Berinde nonexpansive mapping that satisfies condition  $(\bar{I})$ .

**Theorem 3.1.** *Let  $T : C \rightarrow C$  be a Berinde nonexpansive mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$  with  $A(T) \neq \emptyset$  and satisfying condition  $(\bar{I})$ . Suppose that the sequence  $\{x_n\}$  satisfies (2) such that  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ . Then  $\{x_n\}$  converges strongly to  $w \in F(T)$ . Moreover,  $w \in A(T)$ .*

*Proof.* Since  $T$  satisfies condition  $(\bar{I})$ , there exists a nondecreasing function  $h : [0, \infty) \rightarrow [0, \infty)$  with  $h(0) = 0$  and  $h(r) > 0$  for all  $r > 0$  such that

$$(9) \quad 0 \leq h(d(x_n, A(T))) \leq \|x_n - Tx_n\|, \quad \forall n \in \mathbb{N}.$$

It follows from Lemma 3.7 that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0,$$

and hence

$$(10) \quad \lim_{n \rightarrow \infty} h(d(x_n, A(T))) = 0.$$

Since  $h$  is a nondecreasing function satisfying  $h(0) = 0$  and  $h(r) > 0$  for all  $r > 0$ , we get  $\lim_{n \rightarrow \infty} d(x_n, A(T)) = 0$ . Indeed, suppose that  $\lim_{n \rightarrow \infty} d(x_n, A(T)) > 0$ . Then there exist a positive number  $\varepsilon$  and a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  such that

$$d(x_{n_i}, A(T)) > \varepsilon, \quad \forall i \in \mathbb{N}.$$

Therefore,

$$d(x_{n_i}, A(T)) > \frac{\varepsilon}{2}, \quad \forall i \in \mathbb{N},$$

which implies

$$h\left(\frac{\varepsilon}{2}\right) \leq h(d(x_{n_i}, A(T))), \quad \forall i \in \mathbb{N}.$$

Taking limit as  $i \rightarrow \infty$  on both sides and using (10), we get

$$h\left(\frac{\varepsilon}{2}\right) \leq \lim_{i \rightarrow \infty} h(d(x_{n_i}, A(T))) = \lim_{n \rightarrow \infty} h(d(x_n, A(T))).$$

Since  $h$  is a nondecreasing function satisfying  $h(0) = 0$  and  $h(r) > 0$  for all  $r > 0$ , we obtain

$$0 < h\left(\frac{\varepsilon}{2}\right) \leq \lim_{n \rightarrow \infty} h(d(x_n, A(T))) = 0,$$

which is a contradiction. Thus,

$$(11) \quad \lim_{n \rightarrow \infty} d(x_n, A(T)) = 0.$$

Now, we will show that  $\{x_n\}$  is a Cauchy sequence. From (11), for any given  $\epsilon > 0$ , there exists  $m_0 \in \mathbb{N}$  such that

$$d(x_n, A(T)) < \frac{\epsilon}{2}, \quad \forall n \geq m_0.$$

Hence, there is  $v \in A(T)$  such that  $\|x_{m_0} - v\| < \frac{\epsilon}{2}$ . By Lemma 3.6, we know that  $\{\|x_n - v\|\}$  is nonincreasing. Therefore, for any  $m > n \geq m_0$ , we have

$$\begin{aligned} \|x_m - x_n\| &\leq \|x_m - v\| + \|v - x_n\| \\ &\leq \|x_{m_0} - v\| + \|v - x_{m_0}\| \\ &< \epsilon. \end{aligned}$$

This means that  $\{x_n\}$  is a Cauchy sequence in a closed set  $C$ . Thus there exists  $w \in C$  such that

$$(12) \quad \lim_{n \rightarrow \infty} \|x_n - w\| = 0.$$

Using the Berinde nonexpansiveness of  $T$ , we have

$$\begin{aligned} \|w - Tw\| &\leq \|w - x_n\| + \|x_n - Tw\| \\ &\leq \|w - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tw\| \\ &\leq \|w - x_n\| + \|x_n - Tx_n\| + \|x_n - w\| + L\|w - Tx_n\| \\ &\leq \|w - x_n\| + \|x_n - Tx_n\| + \|x_n - w\| + L\|w - x_n\| + L\|x_n - Tx_n\|, \text{ where } L \geq 0. \end{aligned}$$

By using (8) and (12), we obtain  $\|w - Tw\| = 0$ , and hence  $w = Tw$ .

Moreover, applying the condition  $(\bar{I})$  and  $w \in F(T)$ , we find that

$$0 \leq h(d(w, A(T))) \leq \|w - Tw\| = 0.$$

It follows that  $d(w, A(T)) = 0$ , that is,  $w \in A(T)$ . □

An example of a Berinde nonexpansive mapping satisfying condition  $(\bar{I})$  is provided as follow.

**Example 3.4.** Let  $T : [0, 1] \rightarrow [0, 1]$  define as

$$Tx = \begin{cases} x^2 & ; x \in [0, \frac{1}{2}) \\ 0 & ; x \in [\frac{1}{2}, 1]. \end{cases}$$

Then,  $T$  is a Berinde-nonexpansive mapping. Indeed, let  $x, y \in [0, 1]$ .

Case I. If  $x, y \in [\frac{1}{2}, 1]$ , then  $\|Tx - Ty\| = 0 \leq \|x - y\| + \|y - Tx\|$ .

Case II. If  $x, y \in [0, \frac{1}{2})$ , then

$$\begin{aligned} \|Tx - Ty\| &= \|x^2 - y^2\| \\ &\leq \|x - y\| \|x + y\| \\ &\leq \|x - y\| \\ &\leq \|x - y\| + \|y - Tx\|. \end{aligned}$$

Case III. If  $x \in [0, \frac{1}{2})$  and  $y \in [\frac{1}{2}, 1]$ , then  $x^2 \leq \frac{x}{2} \leq x < y$ , and hence

$$\begin{aligned} 2x^2 + x &\leq 2y \\ x^2 &\leq (y - x) + (y - x^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \|Tx - Ty\| &= \|x^2 - 0\| \\ &\leq \|y - x + y - x^2\| \\ &\leq \|y - x\| + \|y - x^2\| \\ &= \|x - y\| + \|y - Tx\|. \end{aligned}$$

Case IV. If  $x \in [\frac{1}{2}, 1]$  and  $y \in [0, \frac{1}{2})$ , similarly to Case III, we have  $\|Tx - Ty\| \leq \|x - y\| + \|y - Tx\|$ .

Moreover,  $T$  satisfies condition  $(\bar{T})$  with  $A(T) = (-\infty, 0]$ . Indeed, for all  $z \in (-\infty, 0]$ .

If  $x \in [0, \frac{1}{2})$ , then

$$\begin{aligned} \|Tx - z\| &= \|x^2 - z\| \\ &\leq \|x - z\|. \end{aligned}$$

If  $x \in [\frac{1}{2}, 1]$ , then  $2z \leq 0 < x$  and hence

$$\begin{aligned} \|Tx - z\| &= \|0 - z\| \\ &\leq \|x - z\|. \end{aligned}$$

Conversely, let  $z \in A(T)$ . Suppose that  $z > 0$ , then we choose  $0 < x < z$  such that  $\|z - Tx\| = \|z - x^2\| > \|z - x\|$  which is a contradiction. This implies that  $A(T) = (-\infty, 0]$ . Finally, we define  $h : [0, +\infty] \rightarrow [0, +\infty]$  such that  $h(r) = r^2$ , then  $h$  is a nondecreasing mapping with  $h(0) = 0$  and  $h(r) > 0$  if  $r > 0$ . We have

$$\|x - Tx\| = \begin{cases} x - x^2 & ; x \in [0, \frac{1}{2}) \\ x & ; x \in [\frac{1}{2}, 1]. \end{cases}$$

Then,

$$h(d(x, A(T))) = x^2 \leq \|x - Tx\|, \forall x \in [0, 1].$$

Therefore,  $T$  is a Berinde nonexpansive mapping satisfying condition  $(\bar{T})$ .

According to Remark 3.3, the iterative process (2) includes the Picard–Mann hybrid iterative process, the Picard–Krasnoselskii hybrid iterative process, and the iterative process (3), the following result is obtained as a direct consequence of Theorem 3.1:

**Corollary 3.1.** *Let  $T : C \rightarrow C$  be a Berinde nonexpansive mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$ , such that  $A(T) \neq \emptyset$ , and suppose that  $T$  satisfies condition  $(\bar{T})$ . Suppose that the sequence  $\{x_n\}$  is generated by one the following iterative processes:*

(i) Mann iterative process:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \gamma_n)x_n + \gamma_nTx_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\gamma_n\} \subset [0, 1]$  satisfies  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ .

(ii) Picard–Mann hybrid iterative process:

$$\begin{cases} x_1 \in C, \\ z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ x_{n+1} = Tz_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\gamma_n\} \subset [0, 1]$  satisfies  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ .

(iii) Picard–Krasnoselskii hybrid iterative process:

$$\begin{cases} x_1 \in C, \\ z_n = (1 - \gamma)x_n + \gammaTx_n, \\ x_{n+1} = Tz_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\gamma \in (0, 1)$  is a constant.

(iv) *Three-step process (as in (3)):*

$$\begin{cases} x_1 \in C, \\ z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ y_n = (1 - \beta_n)z_n + \beta_nTz_n, \\ x_{n+1} = (1 - \alpha_n)Ty_n + \alpha_nTz_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\gamma_n\}, \{\beta_n\}, \{\alpha_n\} \subset [0, 1]$  satisfy  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ .

Then the sequence  $\{x_n\}$  converges strongly to a point  $w \in F(T)$ . Moreover,  $w \in A(T)$ .

Since a nonexpansive mapping is a particular case of Berinde nonexpansive mappings and the iterative process (2) includes the Mann iterative process as a special case, the following is a consequence of Theorem 3.1.

**Corollary 3.2.** *Let  $T : C \rightarrow C$  be a nonexpansive mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$  with  $A(T) \neq \emptyset$  and satisfying condition  $(\bar{I})$ . Suppose that the sequence  $\{x_n\}$  generated by Mann iterative process:*

$$x_{n+1} = (1 - \gamma_n)x_n + \gamma_nTx_n, \text{ for all } n \in \mathbb{N},$$

such that  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ . Then,  $\{x_n\}$  converges strongly to  $w \in F(T)$ . Moreover,  $w \in A(T)$ .

Recall the result of Sensor and Dotson [19] that a mapping  $T : C \rightarrow C$  is said to satisfy condition (I) if there is a nondecreasing function  $h : [0, \infty) \rightarrow [0, \infty)$  with  $h(0) = 0$  and  $h(r) > 0$  for all  $r > 0$  such that

$$0 \leq h(d(x, F(T))) \leq \|x - Tx\|, \quad \forall x \in C,$$

where  $d(x, F(T)) = \inf\{d(x, y) : y \in F(T)\}$ .

Using Corollary 3.2 and Lemma 10 of [29], we have the following as a consequence.

**Theorem 3.2** ([19], Theorem 1.). *Suppose  $X$  is a uniformly convex Banach space,  $C$  is a closed, convex, nonempty subset of  $X$ , and  $T$  is a nonexpansive mapping of  $C$  into  $C$ . Suppose  $T$  satisfies condition (I). Then, for any  $x_1 \in C$ , the sequence  $\{x_n\}$  generated by Mann iterative process:  $x_{n+1} = (1 - \gamma_n)x_n + \gamma_nTx_n$ , for all  $n \in \mathbb{N}$ , where  $\{\gamma_n\} \subset [0, 1]$  satisfies  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ , converges strongly to  $z \in F(T)$ .*

*Proof.* Since  $T$  is nonexpansive and  $F(T) \neq \emptyset$ , we can apply Lemma 10 of [29] to get that  $A(T) \cap C = F(T)$  and hence  $A(T) \neq \emptyset$ . Since  $T$  satisfies the condition (I), there is a nondecreasing function  $h : [0, \infty) \rightarrow [0, \infty)$  with  $h(0) = 0$  and  $h(r) > 0$  for all  $r > 0$  such that

$$0 \leq h(d(x, F(T))) \leq \|x - Tx\|, \quad \forall x \in C.$$

Thus,

$$0 \leq h(d(x, A(T))) \leq h(d(x, A(T) \cap C)) = h(d(x, F(T))) \leq \|x - Tx\|, \quad \forall x \in C,$$

therefore  $T$  satisfies condition  $(\bar{I})$  and hence we apply Corollary 3.2 to complete the proof. □

**Remark 3.4.** Genel and Lindenstrauss [8] showed that there is a nonexpansive mapping on a closed and convex subset of the Hilbert space  $l_2$  such that the sequence defined by Mann iterative process does not converge strongly. Thus, according to Corollary 3.2 the such mapping fails to satisfy condition  $(\bar{I})$ . In other words, without condition  $(\bar{I})$ , the strong convergence can fail in the case of nonexpansive mappings.

**3.4. Strong convergence for contraction mappings.** Observe that the Berinde nonexpansive mapping considered in Theorem 3.1 is required to satisfy *condition*  $(\bar{I})$  in order to ensure convergence. Furthermore, Proposition 3.1 plays a crucial role not only in the proof of Theorem 3.1, but also in the proof of the subsequent result. In the following theorem, we employ the iterative process (2) to establish a strong convergence theorem for the fixed points of a contraction mapping, without the assumption of *condition*  $(\bar{I})$ :

**Theorem 3.3.** *Let  $T$  be a contraction mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$ , and let  $z$  be a fixed point of  $T$ . Suppose that the sequence satisfies (2), i.e., let  $x_1 \in C$ , and*

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \\ x_{n+1} \in S^\infty(z_n), \text{ for all } n \in \mathbb{N}, \end{cases}$$

where  $\{\gamma_n\} \subset [0, 1]$  with  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ . Then  $\{x_n\}$  converges strongly to  $z$ . Moreover  $z \in A(T)$ .

*Proof.* Since  $z \in F(T)$  and  $T$  is a contraction mapping, there exists a constant  $c \in (0, 1)$  such that

$$\|z - Ty\| = \|Tz - Ty\| \leq c\|z - y\| \leq \|z - y\|, \text{ for all } y \in C.$$

Therefore,  $z \in A(T)$ .

From  $x_{n+1} \in S^\infty(z_n)$ ,  $z \in A(T)$ , and Proposition 3.1, we obtain

$$\begin{aligned} \|x_{n+1} - z\| &\leq \|z_n - z\| \\ &= \|(1 - \gamma_n)x_n + \gamma_nTx_n - z\| \\ &\leq (1 - \gamma_n)\|x_n - z\| + \gamma_n\|Tx_n - z\| \\ &= (1 - \gamma_n)\|x_n - z\| + \gamma_n\|Tx_n - Tz\| \\ &\leq (1 - \gamma_n)\|x_n - z\| + \gamma_nc\|x_n - z\| \\ &= (1 - \gamma_n(1 - c))\|x_n - z\|, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Define  $\theta_n := 1 - \gamma_n(1 - c)$ . Then

$$(13) \quad \|x_{n+1} - z\| \leq \theta_n\|x_n - z\|, \text{ for all } n \in \mathbb{N}.$$

Since  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ , there exists a constant  $d \in (0, 1)$  such that

$$\liminf_{n \rightarrow \infty} \gamma_n > \frac{1 - d}{1 - c}.$$

Therefore, there exists  $n_0 \in \mathbb{N}$  such that

$$\inf_{n \geq n_0} \gamma_n > \frac{1 - d}{1 - c},$$

which implies that

$$\gamma_n > \frac{1 - d}{1 - c}, \text{ for all } n \geq n_0.$$

It follows that  $1 - \gamma_n(1 - c) < d$ , for all  $n \geq n_0$ , and hence

$$(14) \quad 0 < \theta_n < d < 1, \text{ for all } n \geq n_0.$$

It follows from (13) and (14) that for all  $n \geq n_0$ ,

$$\begin{aligned} \|x_{n+1} - z\| &\leq \theta_n \|x_n - z\| \\ &\leq \theta_n \theta_{n-1} \|x_{n-1} - z\| \\ &\leq \theta_n \theta_{n-1} \theta_{n-2} \|x_{n-2} - z\| \\ &\vdots \\ &\leq \theta_n \theta_{n-1} \theta_{n-2} \cdots \theta_{n_0} \|x_{n_0} - z\| \\ &< d^{n-n_0+1} \|x_{n_0} - z\|. \end{aligned}$$

From  $0 < d < 1$ , we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - z\| = 0,$$

which implies that  $\{x_n\}$  converges strongly to  $z$ . □

Since the following iterations are included in iterative processes (2), the following result is a direct consequence of Theorem 3.3:

**Corollary 3.3.** *Let  $T : C \rightarrow C$  be a contraction mapping defined on a nonempty closed convex subset  $C$  of a uniformly convex Banach space  $X$ , and let  $z$  be a fixed point of  $T$ . Suppose that the sequence  $\{x_n\}$  is generated by one the following iterative processes:*

- (i) Mann iterative process, where  $\{\gamma_n\} \subset [0, 1]$  with  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ .
- (ii) Picard–Mann hybrid iterative process, where  $\{\gamma_n\} \subset [0, 1]$  with  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ .
- (iii) Picard–Krasnoselskii hybrid iterative process, where  $\gamma \in (0, 1)$  is a constant.
- (iv) Three-step process (as in (3)), where  $\{\gamma_n\}, \{\beta_n\}, \{\alpha_n\} \subset [0, 1]$  and  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ .
- (v) Special case of (2) with  $\gamma_n = 1$ :

$$\begin{cases} x_1 \in C, \\ z_n = Tx_n, \\ x_{n+1} \in S^\infty(z_n), \text{ for all } n \in \mathbb{N}. \end{cases}$$

This follows directly from Theorem 3.3 since  $\gamma_n = 1$  implies  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ .

(vi) Picard iterative process:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Tx_n, \text{ for all } n \in \mathbb{N}. \end{cases}$$

In this case, note that  $x_{n+1} \in S(Tx_n) \subset S^\infty(Tx_n)$ , and thus this iteration is included as a special case of (iv).

Then the sequence  $\{x_n\}$  converges strongly to  $z \in F(T)$ . Moreover  $z \in A(T)$ .

*Proof.* Each of the iterative processes (i)–(iv) satisfies the assumptions of Theorem 3.3 : the mappings are contractions, the control sequences  $\{\gamma_n\}$  satisfy  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ , and the sequence  $\{x_n\}$  satisfies the iterative process (2).

For (v), setting  $\gamma_n = 1$  satisfies  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ , and the structure reduces to a direct application of Theorem 3.3.

For (vi), since  $x_{n+1} = Tx_n$ , it follows that  $x_{n+1} \in S(Tx_n) \subset S^\infty(Tx_n)$ , and hence, it is a special case of (v). Therefore, by (v), the sequence  $\{x_n\}$  converges strongly to  $z \in F(T)$ , and hence  $z \in A(T)$ . □

**3.5. Hilbert-space applications via metric projections.** While nonexpansive mappings guarantee the convexity of the set of fixed points in Hilbert spaces [21], this property is not necessarily guaranteed for Berinde nonexpansive mappings, as in Example 2.1. However, the set of all attractive points for a mapping in Hilbert spaces is convex and closed, as seen in Lemma 2.3.

The following result is motivated by Takahashi and Toyoda [23]:

**Theorem 3.4.** *Let  $T : C \rightarrow C$  be a mapping defined on a nonempty closed convex subset  $C$  of a Hilbert space  $H$  with  $A(T) \cap C \neq \emptyset$ . Let  $P$  be a metric projection from  $C$  onto  $A(T) \cap C$ . Suppose that the sequence  $\{x_n\}$  satisfies (2). Then  $\{Px_n\}$  converges strongly to a fixed point of  $T$ .*

*Proof.* By Lemma 2.3, we have  $A(T)$  is closed and convex in  $H$ . Thus,  $A(T) \cap C$  is closed and convex in  $H$ . By Lemma 3.6, we have

$$\|x_{n+1} - z\| \leq \|x_n - z\|,$$

for all  $n \in \mathbb{N}$  and for all  $z \in A(T) \cap C$ . Applying Lemma 2.5, we obtain  $\{Px_n\}$  converges strongly to some  $v \in A(T) \cap C$ . Moreover, by Remark 2.1, we know that  $A(T) \cap C \subset F(T)$ . Thus,  $v \in F(T)$ . This completes the proof.  $\square$

By applying Theorem 3.1 and Theorem 3.4, we obtain the following result:

**Theorem 3.5.** *Let  $T : C \rightarrow C$  be a Berinde nonexpansive mapping defined on a nonempty closed convex subset  $C$  of a Hilbert space  $H$  with  $A(T) \neq \emptyset$  that satisfies condition  $(\bar{I})$ . Let  $P$  be a metric projection from  $C$  onto  $A(T) \cap C$ . Suppose that the sequence  $\{x_n\}$  satisfies (2) such that  $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$ . Then  $\{x_n\}$  converges strongly to  $z \in F(T)$ , where  $z = \lim_{n \rightarrow \infty} Px_n$ .*

*Proof.* By Theorem 3.1, the sequence  $\{x_n\}$  converges strongly to an element  $w \in F(T)$ , and hence  $w \in A(T)$ . Thus,  $w \in A(T) \cap C$ . Therefore,  $A(T) \cap C \neq \emptyset$ , which implies that a metric projection  $P$  is well-defined. By Theorem 3.4, the sequence  $\{Px_n\}$  converges strongly to an element  $v \in F(T)$ . Since  $P$  is a metric projection of  $C$  onto  $A(T) \cap C$ , we get

$$\langle x_n - Px_n, Px_n - y \rangle \geq 0, \text{ for all } y \in A(T) \cap C.$$

Taking limit as  $n \rightarrow \infty$ , we obtain

$$(15) \quad \langle w - v, v - y \rangle \geq 0, \text{ for all } y \in A(T) \cap C.$$

Choosing  $y = w$  in (15), we have

$$0 \leq \langle w - v, v - w \rangle = -\langle w - v, w - v \rangle = -\|w - v\|^2,$$

which implies  $0 \leq \|w - v\|^2 \leq 0$ . Thus,  $\|w - v\|^2 = 0$ , that is,  $w = v$ .  $\square$

Consequently, the following result is obtained directly from Theorem 3.5:

**Corollary 3.4.** *Let  $T : C \rightarrow C$  be a Berinde nonexpansive mapping defined on a nonempty closed convex subset  $C$  of a Hilbert space  $H$ , such that  $A(T) \neq \emptyset$  and  $T$  satisfies condition  $(\bar{I})$ . Let  $P$  be the metric projection from  $C$  onto  $A(T) \cap C$ . Suppose that the sequence  $\{x_n\}$  is generated by one the following iterative processes:*

(i) *Mann iterative process, where  $\{\gamma_n\} \subset [0, 1]$  satisfies*

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1.$$

(ii) *Picard–Mann hybrid iterative process, where  $\{\gamma_n\} \subset [0, 1]$  satisfies*

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1.$$

- (iii) Picard–Krasnoselskii hybrid iterative process, where  $\gamma \in (0, 1)$  is a constant.  
 (v) Three-step iterative process (as in (3)), where  $\{\gamma_n\} \subset [0, 1]$  satisfies

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1.$$

Then the sequence  $\{x_n\}$  converges strongly to a point  $z \in F(T)$ , where  $z = \lim_{n \rightarrow \infty} Px_n$ .

#### 4. CONCLUSIONS

We proposed a unified iterative scheme (2) which strictly includes many classical fixed point iterative processes. Under mild control conditions and the existence of attractive points in uniformly convex Banach spaces, we showed that all sequences that satisfy the iterative scheme are approximate fixed point sequences. This general result supports strong convergence theorems for Berinde nonexpansive mappings satisfying condition  $(\bar{I})$  and for contraction mappings under minimal control conditions. In Hilbert spaces, combining the scheme with metric projections yields strong convergence to fixed points within the closed convex set of attractive points. Therefore, the proposed scheme offers a flexible and conceptually unified framework that accommodates various iterative algorithms and underscores the importance of attractive points in establishing convergence results.

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