CARPATHIAN J. MATH.
Volume **42** (2026), No. 4,
Pages 755-772

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2026.04.03

Efficient step size rule and golden ratio technique for solving quasimonotone variational inequalities

HAMMED ANUOLUWAPO ABASS[1], AUSTINE EFUT OFEM[2], AND MAGGIE APHANE[3]

ABSTRACT. We propose a unified iterative method that integrates the subgradient extragradient method with the golden ratio technique, tailored for solving quasi-monotone variational inequalities in real Hilbert spaces. The proposed method incorporates a dynamic stepsize rule equipped with both linesearch and self-adaptive techniques, eliminating the need for prior knowledge of the Lipschitz constants. We establish both weak and linear convergence of the proposed method under some mild conditions, extending known results from monotone and pseudo-monotone to quasi-monotone setting. Application to image restoration problem and some numerical examples to demonstrate the efficiency and robustness of our algorithm were discussed.

1. INTRODUCTION

Variational inequality problems (VIPs) have emerged as a fundamental framework for addressing numerous problems in nonlinear analysis, optimization theory, economics, mechanics, and partial differential equations (see [2, 3, 9] and the references therein). Owing to their wide-ranging applications, considerable attention has been given to the study of the existence of solution to this problem in Hilbert spaces.

The variational inequality was initially introduced by Fichera [12], and further developed by Lions and Stampacchia [20] in the context of equilibrium problems. In 1965, Browder [6] extended the theory by formulating variational inequalities in a multi-valued setting. Around the same time, Debrunner and Flor [11] investigated the Minty variational inequality on Banach spaces and established the first known existence results for this formulation.

Let $\mathbb{K}$ be a nonempty closed and convex subset of a real Hilbert space $\mathbb{H}$ and $\mathbb{G}:\mathbb{H}\to\mathbb{H}$ be an operator. Recall that the problem VIP for the operator $\mathbb{G}$ on $\mathbb{K}$ is to find $q\in\mathbb{K}$ such that

$$\langle \mathbb{G}q, p-q\rangle \geq 0 \;\forall\, p\in\mathbb{K}. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathbb{Z}$. The dual VIP is to find $q\in\mathbb{K}$ such that

$$\langle \mathbb{G}p, p-q\rangle \geq 0 \;\forall\, p\in\mathbb{K}, \tag{1.2}$$

we denote the solution set of VIP (1.2) by $\mathbb{Z}_{\mathbb{D}}$. It is obvious that $\mathbb{Z}_{\mathbb{D}}$ is a closed convex set (possibly empty). In the case $\mathbb{G}$ is continuous and $\mathbb{D}$ is convex, we have

$$\mathbb{Z}_{\mathbb{D}} \subset \mathbb{Z}.$$

If $\mathbb{G}$ is a pseudomonotone and continuous, then $\mathbb{Z} = \mathbb{Z}_{\mathbb{D}}$ (see, Lemma 2.1 in [10]). The inclusion $\mathbb{Z}\subset\mathbb{Z}_{\mathbb{D}}$ is false, if $\mathbb{G}$ is quasimonotone and continuous mapping (see, Example 4.2 in [38]).

Received: 16.09.2025. In revised form: 20.12.2025. Accepted: 15.04.2026
2020 *Mathematics Subject Classification.* code, code.
Key words and phrases. *Variational inequality problem; extragradient method; self adaptive stepsize; weak convergence; Hilbert space .*
Corresponding author: Austine Efut Ofem

The most popular projection method for solving variational inequality problem introduced by Korpelevich [18] in finite-dimensional Euclidean spaces, is specifically designed for monotone and Lipschitz continuous operators. This method requires computing two projections onto the feasible set $\mathbb{K}$ per iteration. However, when $\mathbb{K}$ denotes a general closed, convex set, this process may become computationally demanding and challenging, potentially hindering the efficiency of the extragradient method. To dispense with the limitation of the extragradient method, Censor *et al.* [7] proposed the subgradient extragradient method which employs a projection onto half-space instead of the second projection onto $\mathbb{K}$ in the Korpelevich original method. This method can be computed explicitly. Another alternative method to the extragradient method is the well-known method proposed by Tseng [37]. This method requires only one projection per iteration and have been employed for solving variational inequality and variational inclusion problem both in linear and nonlinear spaces. Readers should consult [32, 39, 19, 24, 29] for more details on the methods for solving variational inequalities.

In recent year, the class of pseudomonotone mappings have been studied for solving variational inequality problem (see, [10, 37, 4] and the references therein). In particular when the operator associated with the variational inequality is pseudomonotone and sequentially weakly continuous, the extragradient method is introduced (see [37]). It is worth-mentioning that aforementioned methods in [10, 7] employ pseudomonotone operators. Nonetheless, it is important to mention that the peudomonotonicity of the operator $\mathbb{G}$ cannot be carried over to the case when $\mathbb{G}$ is quasimonotone. For instance, when $\mathbb{G}$ is quasi-monotone, the dual variational inequality is not equivalent to VIP (1.1). Also, the quasi-monotone operators which generalize pseudomonotone operators, have numerous application across various fields due to its flexibility in modeling non-convex, non-smooth or degenerate problems. Also quasi-monotone operators ensure the existence of solution under weaker conditions, making them essential for handling real world models where stronger monotonicity assumptions fail.

It is worth-mentioning that the step-sizes employed in [5, 4, 15, 34, 35] heavily depend on the Lipschitz constant of the operator thereby limiting their applicability. To dispense with the Lipschitz constants, several authors introduced a line search procedure, which is known to consume extra computation time and memory during implementation. As a way to overcome this setback, the self-adaptive technique which is more effective and applicable than the line search procedure was introduced for selecting the stepsize without a prior estimate of the Lipschitz constant.

To accelerate the convergence of iterative processes, various extrapolation techniques have been proposed (see for e.g. [13, 25, 26, 27]). One prominent approach is the inertial method introduced by Polyak [21, 31], which is inspired by a second-order dynamical system designed to enhance convergence speed. This method leverages information from previous iterates by incorporating an additional term, known as the inertial extrapolation term, into the update rule. Another notable contribution is the golden ratio technique, proposed by Malitsky [22], as an extrapolation strategy for solving mixed variational inequalities. This method requires two initial points, which are convexly combined using a fixed weight to generate the next iterate. The weight is determined by the golden ratio, defined as $\beta = \frac{\sqrt{5}+1}{2}$. Owing to its effectiveness in improving convergence behavior, this technique has attracted growing attention in recent research (see [23, 30] and the references therein).

Motivated by the results discussed above, we propose a unified iterative method that integrates the subgradient extragradient method with the golden ratio technique, tailored for solving quasimonotone variational inequalities in real Hilbert spaces. The proposed method incorporates a dynamic stepsize rule equipped with both linesearch and self-adaptive techniques, eliminating the need for prior knowledge of the Lipschitz constants. We establish both weak and linear convergence of the proposed method under some mild conditions. Finally, numerical experiments are presented to illustrate the effectiveness of the new step size procedure. Our findings extend and unify several existing results in the literature. We highlight our contributions as follows:

(1) Our method is tailored to solve variational inequality problem involving a quasi-monotone operator, whereas numerous prior findings are limited to monotone and pseudomonotone operators (see [10, 7, 34, 17]).
(2) The proposed algorithm presented in [17, 10, 15] require prior knowledge of the Lipschitz constant of the operator, which is often computationally demanding, thereby hindering their practical use. In addition, the linesearch methods are known to provide precise approximation of the Lipschitz constant of the operator, but they require longer execution times due to the multiple iteration. Hence, we incorporate a self adaptive stepsize and a linesearch procedure in our proposed method to eliminate the prior information of the Lipschitz contant.
(3) We establish weak and linear convergence of our proposed method. Note that algorithms with linear convergence are more computationally efficient in practice to those that ensure weak convergence. In addition, we include the golden ratio technique to speed up the rate of convergence of our method.
(4) We provide some numerical examples and illustrate the performance over some related ones in the literature. We also apply our result to solve image restoration problem.

The outline of the paper is as follows. In Section 2, definitions and results which are needed for our analysis are presented. Later in Section 3, our proposed method and its convergence analysis was discussed. Then, in Section 4, we present some numerical experiments compare the performances of our proposed method to some related ones in the literature. Lastly, in Section 5, we apply our result to solve an image restoration problem.

2. Preliminaries

In this section, we provide essential definitions, lemmas, and preliminary results that will be used to establish our main convergence theorem. Throughout the paper, we use $\rightharpoonup$ to denote weak convergence and $\to$ to denote strong convergence.

Lemma 2.1. [16] *Let $\mathbb{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Given $x \in \mathbb{H}$ and $z \in \mathbb{K}$. Then, we have*

$$z = P_{\mathbb{K}}x \Leftrightarrow \langle x - z, z - y\rangle \geq 0 \,\forall\, y \in \mathbb{K},$$

where $P_{\mathbb{K}}$ is called the metric projection of $\mathbb{H}$ onto $\mathbb{K}$

Definition 2.2. *Let $\mathbb{G} : \mathbb{H} \to \mathbb{H}$ be a mapping. Then, mapping $\mathbb{G}$ is said to be:*

(1) *$\mathcal{L}$-Lipschitz continuous with $\mathcal{L} > 0$ if*

$$\|\mathbb{G}x - \mathbb{G}y\| \leq \mathcal{L}\|x - y\| \,\forall\, x, y \in H.$$

(2) *monotone, if*

$$\langle \mathbb{G}x - \mathbb{G}y, x - y\rangle \geq 0 \,\forall\, x, y \in H.$$

(3) *pseudomonotone, if*

$$\langle \mathbb{G}x, y - x\rangle \geq 0 \Longrightarrow \langle \mathbb{G}y, y - x\rangle \geq 0 \,\forall\, x, y \in \mathbb{H}. \tag{2.3}$$

(4) *quasimonotone, if*

$$\langle \mathbb{G}x, y - x\rangle > 0 \Longrightarrow \langle \mathbb{G}y, y - x\rangle \geq 0 \; \forall \, x, y \in \mathbb{H}.$$

(5) *δ-strongly pseudomonotone if there exists a constant $\delta > 0$, such that*

$$\langle \mathbb{G}x, x - y\rangle \geq 0 \Longrightarrow \langle \mathbb{G}y, y - x\rangle \geq \delta \|x - y\|^2 \; \forall \, x, y \in \mathbb{H}.$$

(6) *sequentially weakly continuous if, for each sequence $\{x_n\}$ in $\mathbb{H}$, $\{x_k\}$ converges weakly to a point $x \in \mathbb{H}$ implies $\{\mathbb{G}x_k\}$ converges weakly to $\mathbb{G}x$.*
It is easy to see that every (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4), but the converse is not true. The following lemma gives a situation when $\mathbb{Z}_{\mathbb{D}}$ is nonempty.

Lemma 2.3. [39] *If either*

1. *$\mathbb{G}$ is pseudomonotone on $\mathbb{K}$ and $\mathbb{Z} \neq \emptyset$,*
2. *$\mathbb{G}$ is the gradient of $\mathbb{V}$, where $\mathbb{V}$ is a differential quasiconvex function on an open set $\mathbb{C}$, $\mathbb{K} \subset \mathbb{C}$ and attains its global minimum on $\mathbb{K}$,*
3. *$\mathbb{G}$ is quasimonotone on $\mathbb{K}$, $\mathbb{G} \neq 0$, on $\mathbb{K}$ and $\mathbb{K}$ is bounded,*
4. *$\mathbb{G}$ is quasimonotone on $\mathbb{K}$, $\mathbb{G} \neq 0$ on $\mathbb{K}$ and there exists a positive number r such that, for every $v \in \mathbb{K}$ with $\|v\| \geq r$, there exists $y \in \mathbb{K}$ such that $\|y\| \leq r$ and $\langle \mathbb{G}v, y - v\rangle \leq 0$,*
5. *$\mathbb{G}$ is quasimonotone on $\mathbb{K}$, $int\mathbb{K}$ is nonempty and there exists $v^* \in \mathbb{Z}$, such that $\mathbb{G}v^* \neq 0$.*
Then, $\mathbb{Z}_D$ is nonempty.

Lemma 2.4. [14] *Let $\mathbb{K}$ be a nonempty set of $\mathbb{H}$ and $\{x_k\}$ be a sequence in $\mathbb{H}$, such that the following two conditions hold:*

(a) *for all $x \in \mathbb{K}$, $\lim_{k\to\infty} \|x_k - x\|$ exists;*
(b) *every sequential weak cluster point of $\{x_k\}$ is in $\mathbb{K}$.*
Then, $\{x_k\}$ converges to a point in $\mathbb{K}$.

Lemma 2.5. [8] *Assume that the nonnegative sequences $\{\Omega_k\}$ and $\{\Delta_k\}$ satisfy*

$$\Omega_{k+1} \leq \Omega_k - \Delta_k, \; \forall \, k > m,$$

for some nonnegative integer m. Then, $\lim_{k\to\infty} \Delta_k = 0$ and $\lim_{k\to\infty} \Omega_k$ exists.

3. Main Result

To analyze the convergence of the proposed algorithm, we provide the following assumptions:

Assumption 3.1. (A1) $\mathbb{Z}_{\mathbb{D}} \neq \emptyset$,
(A2) $\mathbb{G} : \mathbb{H} \to \mathbb{H}$ is $\mathcal{L}$-Lipschitz continuous on $\mathbb{H}$. However, the information of $\mathcal{L}$ is not necessary to be known.
(A3) $\mathbb{G}$ is sequentially weakly continuous on $\mathbb{K}$, i.e. for each sequence $\{z_k\} \subset \mathbb{K} : \{z_k\}$ converges weakly to q implies that $\{\mathbb{G}z_k\}$ converges weakly to $\mathbb{G}q$.
(A4) $\mathbb{G}$ is quasimonotone on $\mathbb{H}$.

Algorithm 3.2. *Golden ratio technique for quasimonotone variational inequalities.*

Initialization: *Let $z_0, v_1 \in \mathbb{H}$, $\tau, \delta_1 > 0, \ell \in (0, 1), \mu \in (0, 1)$ and $\beta \in (1, +\infty)$ be given.*

Iterative steps: *For each $k \geq 1$, given z_{k-1}, v_k, calculate $\{v_{k+1}\}$ as follows:*

Step 1: *Compute*

$$\begin{cases} z_k = \dfrac{\beta - 1}{\beta} v_k + \dfrac{1}{\beta} z_{k-1} \\ \\ u_k = Proj_{\mathbb{K}}(z_k - \delta_k \mathbb{G}(z_k)). \end{cases} \tag{3.4}$$

If $z_k = u_k$, then stop (z_k is a solution to VIP. Otherwise, go to Step 2.

Step 2: *Construct the half-space*

$$\mathbb{T}_k := \{a \in \mathbb{H} : z_k - \delta_k \mathbb{G}(z_k) - u_k, a - u_k\rangle \leq 0\},$$

and calculate

$$v_{k+1} = Proj_{\mathbb{T}_k}(z_k - \delta_k \mathbb{G}(u_k)), \tag{3.5}$$

and update $\delta_{k+1} = \min\{\delta^{(1)}_{k+1}, \delta^{(2)}_{k+1}\}$, where

(3.6)

$$\delta^{(1)}_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2)}{2\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle}\ ,\ \delta_k \right\} & \textit{if } \langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \neq 0, \\ \delta_k, & \textit{otherwise.} \end{cases}$$

$\delta^{(2)}_{k+1} = \min\{\hat{\delta}^{(2)}_{k+1},\ \delta_k\}$ and $\hat{\delta}^{(2)}_{k+1} = \tau \ell^{m_k}$ with m_k representing the smallest nonnegative integer m such that the following holds:

$$2\tau\ell^m \langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \leq \mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big). \tag{3.7}$$

Stopping criterion *Set $k := k + 1$ and return to* **step 1**.

3.1. **Convergence analysis.** We present the following Lemma for the convergence analysis of Algorithm 3.2.

Lemma 3.3. *Suppose that Assumptions 3.1 holds. Then, the sequence $\{\delta_k\}$ generated by Algorithm 3.2 is non increasing and satisfies*

(i) $\min\{\delta_1, \frac{\mu\ell}{\mathcal{L}}\} \leq \lim_{k\to\infty} \delta_k = \delta \leq \delta_1$.

(ii) $2\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \leq \frac{\mu}{\delta_{k+1}}\|z_k - u_k\|^2 + \frac{\mu}{\delta_{k+1}}\|v_{k+1} - u_k\|^2$.

Proof. We first establish that $\{\delta_k\}$ is non increasing and satisfies the inequality

$$\min\{\delta_1, \frac{\mu\ell}{\mathcal{L}}\} \leq \lim_{k\to\infty} \delta_k = \delta \leq \delta_1.$$

By virtue of (3.6) and (3.7) and the fact that $\delta_{k+1} = \min\{\delta^{(1)}_{k+1},\ \delta^{(2)}_{k+1}\}$, it is evident that $\delta_{k+1} \leq \delta_k$ for all $k \in \mathbb{N}$, and thus $\{\delta_k\}$ is a non-increasing sequence. By the assumption on $\mathbb{G}$ in (3.1), we obtain in the case where $\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \neq 0$, it follows from (3.6) that

$$\begin{aligned} \frac{\mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big)}{2\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle} &\geq \frac{\mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big)}{2\|\mathbb{G}z_k - \mathbb{G}u_k\|\|v_{k+1} - u_k\|} \\ &\geq \frac{\mu\big(\|z_k - u_k\|\|v_{k+1} - u_k\|}{\mathcal{L}\|z_k - u_k\|\|v_{k+1} - u_k\|} \\ &= \frac{\mu}{\mathcal{L}}. \end{aligned}$$

Clearly, $\delta^{(1)}_{k+1} \geq \min\{\delta_k, \frac{\mu}{\mathcal{L}}\}$.

By induction, we obtain the inequality

$$\min\{\delta_1, \frac{\mu}{\mathcal{L}}\} \leq \delta^{(1)}_{k+1} \leq \delta_1. \tag{3.8}$$

By virtue of (3.7), we obtain

$$\frac{2\delta_{k+1}^{(2)}\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle}{\ell} > \mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big).$$

Utilizing the Cauchy-Schwarz inequality, it yields

$$\frac{2\delta_{k+1}^{(2)}}{\ell}\|\mathbb{G}z_k - \mathbb{G}u_k\|\|v_{k+1} - u_k\| > \mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big),$$

Employing the Lipschitz continuity of $\mathbb{G}$, we get

$$\frac{2\delta_{k+1}^{(2)}}{\ell}\mathcal{L}\|z_k - u_k\|\|v_{k+1} - u_k\| > \mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big).$$

Hence,

$$\frac{2\delta_{k+1}^{(2)}}{\ell}\mathcal{L}\frac{\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2}{2} > \mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big),$$

which yields

$$\delta_{k+1}^{(2)} > \frac{\mu\ell}{\mathcal{L}}. \tag{3.9}$$

By combining (3.8) and (3.9), we have

$$\min\Big\{\delta_1, \frac{\mu\ell}{\mathcal{L}}\Big\} \leq \delta_{k+1} \leq \delta_1.$$

Thus, it follows that $\lim_{k\to\infty} \delta_k$ exists and satisfy the given inequality.
(ii) We now establish the inequality in Lemma 3.3 (ii).
From (3.6), we have

$$\delta_{k+1}^{(1)} \leq \frac{\mu\big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\big)}{2\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle},$$

which yields

$$2\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \leq \frac{\mu}{\delta_{k+1}^{(1)}}\|z_k - u_k\|^2 + \frac{\mu}{\delta_{k+1}^{(1)}}\|v_{k+1} - u_k\|^2. \tag{3.10}$$

Similarly, from (3.7) we deduce

$$2\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \leq \frac{\mu}{\delta_{k+1}^{(2)}}\|z_k - u_k\|^2 + \frac{\mu}{\delta_{k+1}^{(2)}}\|v_{k+1} - u_k\|^2. \tag{3.11}$$

By combining (3.10) and (3.11), and using the fact that $\delta_{k+1} = \min\big\{\delta_{k+1}^{(1)}, \delta_{k+1}^{(2)}\big\}$, we obtain

$$2\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \leq \frac{\mu}{\delta_{k+1}}\|z_k - u_k\|^2 + \frac{\mu}{\delta_{k+1}}\|v_{k+1} - u_k\|^2.$$

□

Lemma 3.4. *Let $\{u_k\}$, $\{z_k\}$ and $\{v_{k+1}\}$ be the sequences generated by Algorithm 3.2. Then for all $q \in \mathbb{Z}$, we have*

$$\|v_{k+1} - q\|^2 \leq \|z_k - q\|^2 - \big(1 - \mu\frac{\delta_k}{\delta_{k+1}}\big)\Big(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\Big),$$

and thus

$$\|v_{k+1} - q\| \leq \|z_k - q\|.$$

Proof. Since $q \in \mathbb{Z}_{\mathbb{D}} \subset \mathbb{Z} \subset \mathbb{K} \subset \mathbb{T}_k$, we have

$$\begin{aligned}\|w_k - q\|^2 &= \|Proj_{\mathbb{T}_k}(z_k - \delta_k \mathbb{G}(u_k)) - Proj_{T_k} q\|^2 \\ &\leq \langle v_{k+1} - q, z_k - \delta_k \mathbb{G}(u_k) - q\rangle \\ &= \frac{1}{2}\|v_{k+1} - q\|^2 + \frac{1}{2}\|z_k - \delta_k \mathbb{G}(u_k) - q\|^2 \\ &- \frac{1}{2}\|v_{k+1} - z_k + \delta_k \mathbb{G}(u_k)\|^2 \\ &= \frac{1}{2}\|v_{k+1} - q\|^2 + \frac{1}{2}\|z_k - q\|^2 + \frac{1}{2}\delta_k^2\|\mathbb{G}(u_k)\|^2 \\ &- \langle z_k - q, \delta_k \mathbb{G}(u_k)\|^2 - \langle w_k - z_k, \delta_k \mathbb{G}(u_k)\rangle \\ &= \frac{1}{2}\|v_{k+1} - q\|^2 + \frac{1}{2}\|z_k - q\|^2 - \frac{1}{2}\|w_k - z_k\|^2 \\ &- \langle w_k - q, \delta_k \mathbb{G}(u_k)\rangle.\end{aligned}$$

This implies

$$\|v_{k+1} - q\|^2 \leq \|z_k - q\|^2 - \|w_k - z_k\|^2 - 2\langle w_k - q, \delta_k \mathbb{G}(u_k)\rangle. \tag{3.12}$$

Since $q \in \mathbb{Z}_{\mathbb{D}}$ and $u_k \in \mathbb{K}$, we have

$$\langle \mathbb{G}u_k, q - u_k\rangle \leq 0.$$

Thus, we have

$$\begin{aligned}\langle \mathbb{G}u_k, q - w_k\rangle &= \langle \mathbb{G}u_k, q - u_k\rangle + \langle \mathbb{G}u_k, u_k - w_k\rangle \\ &\leq \langle \mathbb{G}u_k, u_k - w_k\rangle.\end{aligned} \tag{3.13}$$

By virtue of (3.12) and (3.13), we have

$$\begin{aligned}\|v_{k+1} - q\|^2 &\leq \|z_k - q\|^2 - \|v_{k+1} - z_k\|^2 + 2\delta_k\langle \mathbb{G}u_k, u_k - v_{k+1}\rangle \\ &= \|z_k - q\|^2 - \|v_{k+1} - u_k\|^2 - \|u_k - z_k\|^2 \\ &- 2\langle v_{k+1} - u_k, u_k - z_k\rangle + 2\delta_k\langle \mathbb{G}u_k, u_k - v_{k+1}\rangle \\ &= \|z_k - q\|^2 - \|v_{k+1} - u_k\|^2 - \|u_k - z_k\|^2 \\ &+ 2\langle z_k - \delta_k \mathbb{G}u_k - u_k, v_{k+1} - u_k\rangle.\end{aligned} \tag{3.14}$$

It is obvious from the definition of u_k and $\mathbb{T}_k$ that $u_k = Proj_{\mathbb{T}_k}(z_k - \delta_k \mathbb{G}z_k)$ and since $w_k \in \mathbb{T}_k$, we have

$$2\langle u_k + \delta_k \mathbb{G}z_k - z_k, u_k - v_{k+1}\rangle \leq 0. \tag{3.15}$$

From (3.15), we deduce

$$2\langle z_k - \delta_k \mathbb{G}z_k - u_k, v_{k+1} - u_k\rangle \leq 0. \tag{3.16}$$

By utilizing (3.16) and Lemma 3.3 (ii), we get

$$\begin{aligned}2\langle z_k - \delta_k \mathbb{G}u_k - u_k, v_{k+1} - u_k\rangle &= 2\langle z_k - \delta_k \mathbb{G}z_k - u_k, v_{k+1} - u_k\rangle \\ &+ 2\delta_k\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \\ &\leq 2\delta_k\langle \mathbb{G}z_k - \mathbb{G}u_k, v_{k+1} - u_k\rangle \\ &\leq \mu\frac{\delta_k}{\delta_{k+1}}\|z_k - u_k\|^2 + \mu\frac{\delta_k}{\delta_{k+1}}\|v_{k+1} - u_k\|^2.\end{aligned} \tag{3.17}$$

On substituting (3.17) into (3.13), it yields

$$\|v_{k+1}-q\|^2 \leq \|z_k-q\|^2 - \big(1-\mu\frac{\delta_k}{\delta_{k+1}}\big)\|z_k-u_k\|^2 - \big(1-\mu\frac{\delta_k}{\delta_{k+1}}\big)\|v_{k+1}-u_k\|^2. \tag{3.18}$$

Now, since $\lim_{k\to\infty}\big(1-\mu\frac{\delta_k}{\delta_{k+1}}\big) = 1-\mu > 0$. It follows that there exists $k_0 \in \mathbb{N}$ such that $1-\mu\frac{\delta_k}{\delta_{k+1}} > 0,\ \forall\, k \geq k_0$. Thus, we obtain

$$\|v_{k+1}-q\| \leq \|z_k-q\|\ \forall\, k \geq k_0. \tag{3.19}$$

□

Lemma 3.5. *Let $\{z_k\}$ be the sequence defined by Algorithm 3.2 and suppose that Assumption 3.1 holds. If there exists a subsequence $\{z_{k_l}\}$ which converges weakly to $p \in \mathbb{H}$ and $\lim_{l\to\infty}\|z_{k_l}-u_{k_l}\| = 0$, then $p \in \mathbb{Z}_{\mathbb{D}}$ of $\mathbb{G}z = 0$.*

Proof. Since $\{z_{k_l}\} \rightharpoonup p$ and $\lim_{l\to\infty}\|z_{k_l}-u_{k_l}\| = 0$, this implies that $\{u_{k_l}\} \rightharpoonup p$ and since $u_k \in \mathbb{K}$, we obtain that $p \in \mathbb{K}$. Now, we divide the proof into two cases:
Case 1: If $\limsup_{l\to\infty}\|\mathbb{G}u_{k_l}\| = 0$, then we have $\lim_{l\to\infty}\|\mathbb{G}u_{k_l}\| = \liminf_{l\to\infty}\|\mathbb{G}u_{k_l}\| = 0$. Since $u_{k_l} \rightharpoonup p \in \mathbb{K}$ and $\mathbb{G} \rightharpoonup p \in \mathbb{K}$ and $\mathbb{G}$ satisfies (A3), we get

$$0 \leq \|\mathbb{G}p\| \leq \liminf_{l\to\infty}\|\mathbb{G}u_{k_l}\| = 0.$$

Thus $\mathbb{G}p = 0$.
Case 2: If $\limsup_{l\to\infty}\|Gu_{k_l}\| > 0$. Without loss of generality, we take $\lim_{l\to\infty}\|\mathbb{G}u_{k_l}\| = M > 0$. It then follows that there exists a $k \in \mathbb{N}$, such that $\|\mathbb{G}u_{k_l}\| > \frac{M}{2}$ for all $k \geq \mathbb{N}$. Since $u_{k_l} = Proj(z_{k_l}-\delta_{k_l}\mathbb{G}z_{k_l})$, we get

$$\langle z_{k_l}-\delta_{k_l}\mathbb{G}z_{k_l}-u_{k_l}, a-u_{k_l}\rangle \leq 0,\ \forall\, a \in \mathbb{K},$$

or equivalently

$$\frac{1}{\delta_{k_l}}\langle z_{k_l}-u_{k_l}, a-u_{k_l}\rangle \leq \langle \mathbb{G}z_{k_l}, a-u_{k_l}\rangle,\ \forall\, a \in \mathbb{K}.$$

Consequently, we have

$$\frac{1}{\delta_{k_l}}\langle z_{k_l}-u_{k_l}, a-u_{k_l}\rangle + \langle \mathbb{G}z_{k_l}, u_{k_l}-z_{k_l}\rangle \leq \langle \mathbb{G}z_{k_l}, a-z_{k_l}\rangle,\ \forall\, a \in \mathbb{K}. \tag{3.20}$$

Since $\{z_{k_l}\}$ is weakly convergent and bounded. Utilizing the Lipschitz continuity of $\mathbb{G}$, $\{\mathbb{G}z_{k_l}\}$ is bounded. As $\|z_{k_l}-u_{k_l}\| \to 0$, $\{u_{k_l}\}$ is also bounded and $\delta_{k_l} \geq \min\{\delta_1, \frac{\mu}{\mathcal{L}}\}$. By passing limit as $k \to \infty$ on (3.20), we get

$$\liminf_{l\to\infty}\langle \mathbb{G}z_{k_l}, a-z_{k_l}\rangle \geq 0,\ \forall\, a \in \mathbb{K}, \tag{3.21}$$

moreover, we have

$$\langle \mathbb{G}u_{k_l}, a-u_{k_l}\rangle = \langle \mathbb{G}u_{k_l}-\mathbb{G}z_{k_l}, a-z_{k_l}\rangle + \langle \mathbb{G}z_{k_l}, a-z_{k_l}\rangle + \langle \mathbb{G}u_{k_l}, z_{k_l}-u_{k_l}\rangle. \tag{3.22}$$

Since $\lim_{l\to\infty}\|z_{k_l}-u_{k_l}\| = 0$ and $\mathbb{G}$ is $\mathcal{L}$-Lipschitz continuous on $\mathbb{H}$, we obtain

$$\lim_{l\to\infty}\|\mathbb{G}z_{k_l}-\mathbb{G}u_{k_l}\| = 0.$$

This implies from (3.21) and (3.22) that

$$\lim_{l\to\infty}\langle \mathbb{G}u_{k_l}, a-u_{k_l}\rangle \geq 0. \tag{3.23}$$

If $\limsup_{l\to\infty}\langle \mathbb{G}u_{k_l}, a-u_{k_l}\rangle > 0$, then there exists a subsequence $\{u_{k_{l_j}}\}$ such that $\lim_{j\to\infty}\langle \mathbb{G}u_{k_{l_j}}, a-u_{k_{l_j}}\rangle > 0$. Consequently, there exists $j_0 \in \mathbb{N}$ such that

$$\langle \mathbb{G}u_{k_{l_j}}, a-u_{k_{l_j}}\rangle > 0 \;\forall\, j \geq j_0.$$

By letting $j \to \infty$, we have $p \in \mathbb{Z}_{\mathbb{D}}$.
If $\limsup_{l\to\infty}\langle \mathbb{G}u_{k_l}, a-u_{k_l}\rangle = 0$. This implies from (3.23) that

$$\lim_{l\to\infty}\langle \mathbb{G}u_{k_l}, a-u_{k_l}\rangle = 0.$$

Let

$$\xi_k := \left|\langle \mathbb{G}u_{k_l}, a-u_{k_l}\rangle\right| + \frac{1}{k+1}.$$

We obtain that

$$\langle \mathbb{G}u_{k_l}, a-u_{k_l}\rangle + \xi_k > 0,\; \forall\, l \geq K. \tag{3.24}$$

In addition, for each $l \geq 1$, since $\{u_{k_l}\} \subset \mathbb{K}$, we assume that $\mathbb{G}u_{k_l} \neq 0$ (otherwise, u_{k_l} is a solution). Now, setting

$$b_{k_l} = \frac{\mathbb{G}u_{k_l}}{\|\mathbb{G}u_{k_l}\|^2},$$

we have $\langle \mathbb{G}u_{k_l}, b_{k_l}\rangle = 1$ for each $l \geq K$. We can easily deduce from (3.24) that

$$\langle \mathbb{G}u_{k_l}, a+\xi_l b_{k_l}-u_{k_l}\rangle > 0.$$

Utilizing (A4), we get

$$\begin{aligned}
\big\langle \mathbb{G}a, a+\xi_l b_{k_l}-u_{k_l}\big\rangle &= \big\langle \mathbb{G}a - \mathbb{G}(a+\xi_l b_{k_l}), a+\xi_l b_{k_l}-u_{k_l}\big\rangle \\
&\quad + \big\langle \mathbb{G}(a+\xi_l b_{k_l}), a+\xi_l b_{k_l}-u_{k_l}\big\rangle \\
&\geq \big\langle \mathbb{G}a - \mathbb{G}(a+\xi_l b_{k_l}), a+\xi_l b_{k_l}-u_{k_l}\big\rangle \\
&\geq -\|\mathbb{G}a - \mathbb{G}(a+\xi_l b_{k_l})\|\,\|a+\xi_l b_{k_l}-u_{k_l}\| \\
&\geq -\xi_l\mathcal{L}\|b_{k_l}\|\,\|a+\xi_l b_{k_l}-u_{k_l}\| \\
&= -\xi_l\mathcal{L}\frac{1}{\|\mathbb{G}u_{k_l}\|}\|a+\xi_l b_{k_l}-u_{k_l}\| \\
&\geq -\xi_l\mathcal{L}\frac{2}{M}\|a+\xi_l b_{k_l}-u_{k_l}\|. \qquad (3.25)
\end{aligned}$$

By passing limit as $k \to \infty$ on (3.25) and utilizing $\lim_{l\to\infty}\xi_l = 0$ and the boundedness of $\{\|a+\xi_l b_{k_l}-u_{k_l}\|\}$, we get

$$\langle \mathbb{G}a, a-p\rangle \geq 0,\; \forall\, a \in \mathbb{K}.$$

This implies that $p \in \mathbb{Z}_{\mathbb{D}}$. □

Theorem 3.6. *Let $\{v_k\}$ be the sequence generated by Algorithm 3.2 such that Assumption 3.2 are satisfied. Then $\{v_k\}$ is bounded and converges weakly to $q \in \mathbb{Z}_{\mathbb{D}} \subset \mathbb{Z}$.*

Proof. From step 1 of Algorithm 3.2, we have

$$v_k = \frac{\beta}{\beta-1} z_k - \frac{1}{\beta-1} z_{k-1}.$$

Thus

$$\begin{aligned}\|v_k - q\|^2 &= \left\| \frac{\beta}{\beta-1} z_k - \frac{1}{\beta-1} z_{k-1} - q \right\|^2 \\ &= \frac{\beta}{\beta-1}\|z_k - q\|^2 - \frac{1}{\beta-1}\|z_{k-1} - q\|^2 + \frac{\beta}{(\beta-1)^2}\|z_k - z_{k-1}\|^2 \\ &= \frac{\beta}{\beta-1}\|z_k - q\|^2 - \frac{1}{\beta-1}\|z_{k-1} - q\|^2 + \frac{1}{\beta}\|v_k - z_{k-1}\|^2. \end{aligned} \tag{3.26}$$

By virtue of (3.18) and (3.26), we have

$$\begin{aligned}\|v_{k+1} - q\|^2 - \|v_k - q\|^2 \le{}& \frac{-1}{\beta-1}\|z_k - q\|^2 + \frac{1}{\beta-1}\|z_{k-1} - q\|^2 - \frac{1}{\beta}\|v_k - z_{k-1}\|^2 \\ &- \left(1 - \mu\frac{\delta_k}{\delta_{k+1}}\right)\left(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\right),\end{aligned}$$

which also yields

$$\begin{aligned}\|v_{k+1} - q\|^2 + \frac{1}{\beta-1}\|z_k - q\|^2 \le{}& \|v_k - q\|^2 + \frac{1}{\beta-1}\|z_{k-1} - q\|^2 - \frac{1}{\beta}\|v_k - z_{k-1}\|^2 \\ &- \left(1 - \mu\frac{\delta_k}{\delta_{k+1}}\right)\left(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\right).\end{aligned} \tag{3.27}$$

Let

$$\Phi_k = \|v_k - q\|^2 + \frac{1}{\beta-1}\|z_{k-1} - q\|^2,$$

and

$$\Delta_k = \frac{1}{\beta}\|v_k - z_{k-1}\|^2 + \left(1 - \mu\frac{\delta_k}{\delta_{k+1}}\right)\left(\|z_k - u_k\|^2 + \|v_{k+1} - u_k\|^2\right),$$

we infer that

$$\Phi_{k+1} \le \Phi_k - \Delta_k.$$

We also have $\Phi_{k+1} \ge 0$ and $\Delta_k \ge 0$. By Lemma 2.5. we now see that $\lim_{k\to\infty} \Delta_k = 0$ and that $\lim_{k\to\infty} \Phi_k$ exists. Hence, we obtain

$$\lim_{k\to\infty} \|v_{k+1} - u_k\| = 0 = \lim_{k\to\infty} \|z_k - u_k\|, \tag{3.28}$$

and that

$$\lim_{k\to\infty} \|v_k - z_{k-1}\| = 0. \tag{3.29}$$

Using the definition of z_k, we have

$$
\begin{aligned}
\Phi_{k+1} &= \|v_{k+1}-q\|^2 + \frac{1}{\beta-1}\|z_k-q\|^2 \\
&= \frac{\beta}{\beta-1}\|z_{k+1}-q\|^2 + \frac{\beta}{(\beta-1)^2}\|z_{k+1}-z_k\|^2 - \frac{1}{\beta-1}\|z_k-q\|^2 \\
&+ \frac{1}{\beta-1}\|z_k-q\|^2 \\
&= \frac{\beta}{\beta-1}\|z_{k+1}-q\|^2 + \frac{\beta}{(\beta-1)^2}\|z_{k+1}-z_k\|^2 \\
&= \frac{\beta}{\beta-1}\|z_{k+1}-q\|^2 + \frac{1}{\beta}\|v_{k+1}-z_k\|^2.
\end{aligned}
$$

It is easy to see that the limit of $\{\|z_{k+1}-q\|^2\}$ exists. We also conclude that $\lim_{k\to\infty}\|v_{k+1}-q\|^2$ exists and that the sequences $\{v_k\}$ and $\{z_k\}$ are bounded. Consequently, $\{u_k\}$ is also bounded. Since $\lim_{k\to\infty}\|v_k-q\|$ exists and each sequentially weak cluster point of the sequence $\{v_k\}$ is in $\mathbb{Z}_{\mathbb{D}} \subset \mathbb{Z}$. By Lemma 2.4, $\{v_k\}$ converges weakly to an element of $\mathbb{Z}_{\mathbb{D}} \subset \mathbb{Z}$. □

Theorem 3.7. *Let $\{v_k\}$ be the sequence generated by Algorithm 3.2 such that Assumption 3.1 are satisfied and the mapping $\mathbb{G}$ is $\rho-$ strongly pseudomonotone with $\rho > 0$. Then, $\{v_k\}$ R-linearly converges to the unique point $q \in \mathbb{Z}_{\mathbb{D}} \subset \mathbb{Z}$.*

Proof. From the strong monotonicity property of $\mathbb{G}$ and since $u_k \in \mathbb{K}$, we obtain

$$\langle \mathbb{G}u_k, q-u_k\rangle \le -\rho\|u_k-q\|^2.$$

Since $\delta_k > 0$, we get

$$\delta_k\langle \mathbb{G}u_k, q-u_k\rangle \le -\delta_k\rho\|u_k-q\|^2.$$

Observe that

$$
\begin{aligned}
\delta_k\langle \mathbb{G}u_k, q-v_{k+1}\rangle &= \delta_k\langle \mathbb{G}u_k, q-u_k\rangle + \langle \mathbb{G}u_k, u_k-v_{k+1}\rangle \\
&\le -\delta_k\rho\|u_k-q\|^2 + \langle \mathbb{G}u_k, u_k-v_{k+1}\rangle. \qquad (3.30)
\end{aligned}
$$

Using (3.30) and the proof in Lemma 3.3, we get

$$
\begin{aligned}
\|v_{k+1}-q\|^2 &\le \|z_k-q\|^2 - \left(1-\mu\frac{\delta_k}{\delta_{k+1}}\right)\|v_{k+1}-u_k\|^2 \\
&\quad - \left(1-\mu\frac{\delta_k}{\delta_{k+1}}\right)\|z_k-u_k\|^2 - 2\delta_k\rho\|u_k-q\|^2 \\
&\le \|z_k-q\|^2 - \left(1-\mu\frac{\delta_k}{\delta_{k+1}}\right)\|z_k-u_k\|^2 - 2\delta_k\rho\|u_k-q\|^2. \qquad (3.31)
\end{aligned}
$$

Note that

$$
\begin{aligned}
-\left(1-\mu\frac{\delta_k}{\delta_{k+1}}\right)\|z_k-u_k\|^2 - 2\delta_k\rho\|u_k-q\|^2 &\le -\min\left\{(1-\frac{\mu\delta_k}{\delta_{k+1}}),\ 2\delta_k\rho\right\}\Big(\|z_k-u_k\|^2 \\
&\quad + \|u_k-q\|^2\Big) \\
&\le -\min\left\{\frac{1}{2}(1-\frac{\mu\delta_k}{\delta_{k+1}}),\ \delta_k\rho\right\}\|z_k-q\|^2 \\
&\le -\min\left\{\frac{1}{2}(1-\eta),\ \frac{1}{2}\delta\rho\right\}\|z_k-q\|^2,\ \forall\, k \ge k_0. \qquad (3.32)
\end{aligned}
$$

Equation (3.32) holds since there exists $k_0 \in \mathbb{N}$ such that $\delta_k > \dfrac{\delta}{2}$ and $\big(1 - \dfrac{\mu\delta_k}{\delta_{k+1}}\big) > (1 - \eta)\ \forall\ k \geq k_0$. On substituting this estimate into (3.32), it yields

$$\|v_{k+1} - q\|^2 \leq \left(1 - \min\left\{\frac{1}{2}(1-\eta),\ \frac{1}{2}\delta\rho\right\}\right)\|z_k - q\|^2. \tag{3.33}$$

From $(1-\eta) < \big(1 - \dfrac{\mu\delta_k}{\delta_{k+1}}\big)$, it follows that $0 < \gamma^2 < 1$, where

$$\gamma^2 = 1 - \min\left\{\frac{1}{2}(1-\eta),\ \frac{1}{2}\delta\rho\right\}$$

Thus, we have

$$\|v_{k+1} - q\|^2 \leq \gamma^2\|z_k - q\|^2,\ \forall\ k \geq k_0. \tag{3.34}$$

Utilizing (3.26), we get

$$\frac{\beta}{\beta-1}\|z_{k+1} - q\|^2 \leq \big(\gamma^2 + \frac{1}{\beta-1}\big)\|z_k - q\|^2,\ \forall\ k \geq k_0.$$

Since $0 \leq \gamma^2 < 1$, we get $\gamma + \dfrac{1}{\beta-1} < 1 + \dfrac{1}{\beta-1}$, which implies that

$$0 < \frac{\gamma^2 + \dfrac{1}{\beta-1}}{\dfrac{\beta}{\beta-1}} < 1.$$

Hence,

$$\|z_{k+1} - q\|^2 < \delta\|z_k - q\|^2,\ \forall\ k \geq k_0,$$

By mathematical induction, we have

$$\|z_{k+1} - q\|^2 < \delta^{(k+1)}\|z_0 - q\|^2, \tag{3.35}$$

which in view of the fact that $0 < \delta < 1$, implies

$$\lim_{k\to\infty}\|z_{k+1} - q\| = 0.$$

Then, by (3.34), we directly get that

$$\lim_{k\to\infty}\|v_{k+1} - q\| = 0.$$

Thus, we conclude that $\{v_k\}$ linearly converges to the unique point $q \in \mathbb{Z}_{\mathbb{D}} \subset \mathbb{Z}$. □

4. Numerical Example

In this section, we present some numerical examples to illustrate the efficiency of our suggested Algorithm 3.2 when compared with Algorithm 3.2 of Alakoya et al. [1] (shortly, OMS Alg 3.2), Algorithm 3.1 of Thong et al. [33] (shortly, TLDTL Alg 3.1) and Algorithm 3.1 of Ofem at al. [24] (shortly, OMUIN Alg 3.1). All the numerical computations were conducted using the MATLAB version R2025a and the stopping criterion $\|v_{k+1} - v_k\| < 10^{-5}$.

In the numerical results in this section, "Iter." and "Sec." denote the number of iterations and the CPU time in seconds, respectively. Take $\tau = 0.1$, $\delta_1 = 0.9$, $\ell = 0.5$, $\mu = 0.8$ and $\beta = 2$ for our algorithm (3.2). Choose $\tau_1 = 0.5$, $\alpha = 0.4$, $\beta = 0$, $\lambda = 0.1\frac{(1-\alpha)^2}{(1-\alpha)^2 + \max\{\alpha(1+\alpha), \beta(1+\beta)\}}$, $\mu = 0.9$ and $\alpha_n = \frac{1}{n^2}$ for TLDTL Alg 3.1. Choose $\theta_1 = 0.7$,

$\lambda_1 = 1$, $\mu = 0.3$, and $\alpha_n = \frac{1}{k^2}$ for MNO Alg 3.2 and take $\tau = 0.6$, $\lambda_1 = 0.6$, $\gamma_m = \theta_m = \frac{1}{(m+1)^2}$ for OMUIN Alg 3.1.

Example 4.1. [19] *Let $\mathbb{K} = [-1, 1]$ and the operator $\mathbb{G} : \mathbb{K} \to \mathbb{K}$ be dfined by*

$$\mathbb{G} = \begin{cases} 2v - 1, & v > 1, \\ v^2, & v \in [-1, 1], \\ -2v - 1, & v < -1. \end{cases} \tag{4.36}$$

Then, $\mathbb{G}$ is quasimonotone and Lipschitz continuous with $\mathbb{Z}_{\mathbb{D}} = \{-1\}$ and $\mathbb{Z} = \{-1, 0\}$.

We choose the initial values v_0 and v_1 as follows:
Case a: $v_0 = 0.9$ *and* $v_1 = -1.0$,
Case b: $v_0 = 0.8$ *and* $v_1 = -1.0$,
Case c: $v_0 = 0.7$ *and* $v_1 = -1.0$,
Case d: $v_0 = 0.6$ *and* $v_1 = -1.0$.

TABLE 1. Numerical results of Example 4.1

Cases		*Algorithm 3.2*	*OMUIN Alg 3.1*	*OMS Alg 3.2*	*TLDTL 3.1*
Case 1	*CPU time (sec.)*	*0.4335*	*0.6054*	*1.0484*	*1.0984*
	No of Iter.	*220*	*380*	*1010*	*1850*
Case 2	*CPU time (sec.)*	*0.6549*	*0.9859*	*2.0380*	*3.9503*
	No of Iter.	*550*	*1020*	*3050*	*5990*
Case 3	*CPU time (sec.)*	*0.4755*	*0.6314*	*1.0664*	*1.1957*
	No of Iter.	*230*	*380*	*1015*	*1860*
Case 4	*CPU time (sec.)*	*0.4889*	*0.9967*	*2.0877*	*3.9917*
	No of Iter.	*555*	*1024*	*3055*	*5995*

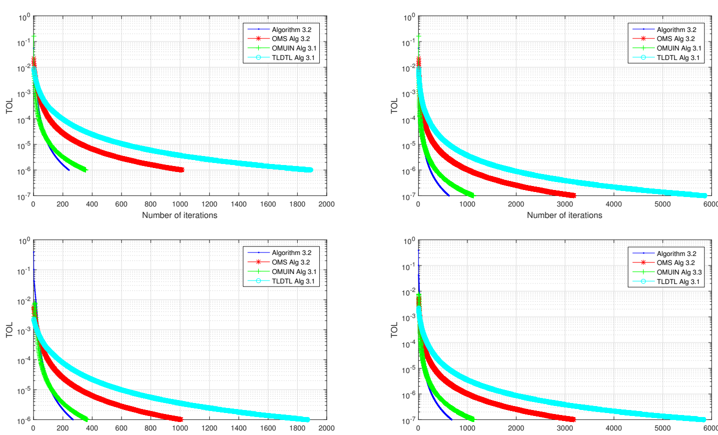

FIGURE 1. Example 4.1, **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

Example 4.2. [19] *Let $\mathbb{K} = [0, 1]^m$ and $\mathbb{G}v = (u_1 v, u_2 v, \cdots, u_m v)$, where $u_i v = v_{i-1}^2 + v_i^2 + v_{i-1}v_i + v_i v_{i+1} - 2v_{i-1} + 4v_i + v_{i+1} - 1$, $i = 1, 2, \cdots, m$, $u_0 = u_{m+1} = 0$. Then $\mathbb{G}$ is quasimonotone.*

For this numerical experiment, we choose $m \in \{40, 80, 100, 140\}$ *and then randomly generate* v_0 *and* v_1.

TABLE 2. Results of the Numerical Simulations for Different Dimensions

Numerical Results for $m = 40, 80, 100$ *and* 140 *in Example 4.2*								
	Algorithm 3.2		*OMUIN Alg 3.1*		*OMS Alg 3.2*		*TLDTL Alg 3.1*	
m	*Iter*	*CPU time (sec.)*	*Iter*	*CPU time (sec.)*	*Iter*	*CPU time (sec.)*	*Iter*	*CPU time (sec.)*
40	48	0.0522	64	0.0662	100	0.7241	110	1.0418
80	50	0.0628	66	0.0787	105	0.8399	111	1.0554
100	50	0.0655	60	0.0541	110	0.9671	150	1.2632
140	60	0.0889	80	0.0882	125	0.9983	185	1.0946

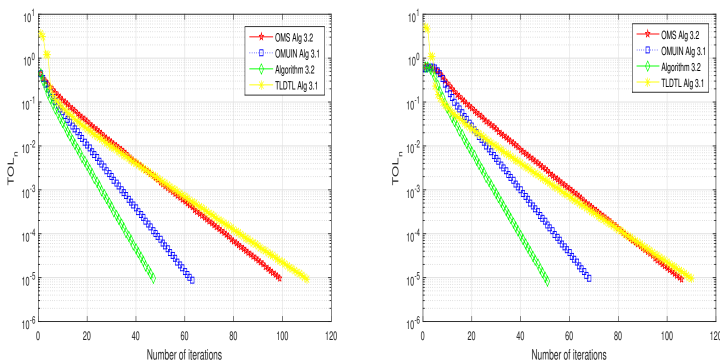

FIGURE 2. Graph of the Iterates for Example 4.2 when the Dimensions $m = 40$ and $m = 80$

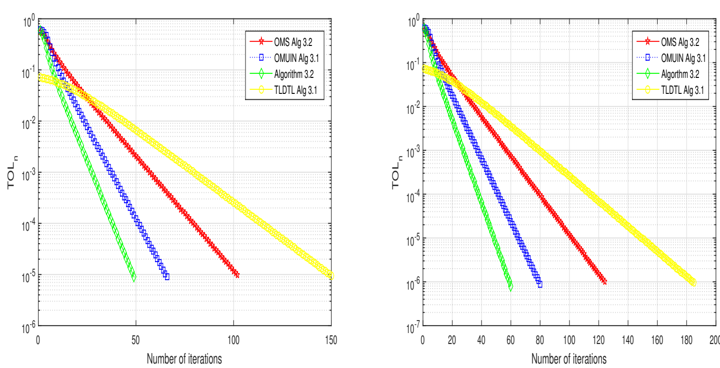

FIGURE 3. Graph of the Iterates for Example 4.2 when the Dimensions $m = 100$ and $m = 140$

5. APPLICATION TO MONOTONE INCLUSION PROBLEM

In this section, we apply our result to image restoration problem. We test the efficiency of our Algorithm 3.2 with Algorithms Algorithm 3.3 Mewomo et al. [1] (shortly, MNO

Alg 3.3), Algorithm 3.1 of Thong et al. [33] (shortly, TLDTL Alg 3.1) and Algorithm 3.1 of Ofem at al. [24] (shortly, OMUIN Alg 3.1). The image restoration problem can be formulated as the following linear inverse problem:

$$x = Dv + e \tag{5.37}$$

where $D \in \mathbb{R}^{M\times N}$ stand for the blurring matrix, $x \in \mathbb{R}^{M\times N}$ is the observed blurred image, $v \in \mathbb{R}^{M\times N}$ repents the original image and e is the Gaussian noise. It is worthy to note that solving Problem (5.37) is equivalent to solving the convex minimization problem

$$\min_{v\in\mathbb{R}^N} \{\frac{1}{2}||Dv - x||_2^2 + \lambda||v||_1\} \tag{5.38}$$

where $\|\cdot\|$ stand for the Euclidean norm, $\|\cdot\|_1$ denotes the ℓ_1-norm and where $\lambda > 0$ is the regularization parameter. Our aim here is to recover the original image v given the data of the blurred image x. One can express the minimization problem (5.38) as a variational inequality problem by taking $\mathbb{G} = D^T(Dv - x)$. It is well known $\mathbb{G}$ is monotone and $\|D^T D\|$-Lipschitz continuous. For this experiment, we consider the 291 × 240 tire image from MATLAB Image Processing Toolbox. furthermore, we employ the Gaussian blur of size 9 × 9 and standard deviation $\sigma = 7$ to formulate the blurred and noisy image (observed image) and use the various algorithms to restore the original image from its blurred form. Again, we measure the quality of the recovered image through the signal-to-noise ratio defined by

$$SNR = 30\log_{10}\frac{\|v\|_2}{\|v - v^*\|} \tag{5.39}$$

where v^* is the restored image and v is the original image. Interestingly, the larger the SNR, the better the quality of the recovered image. Now, we take the starting values as $v_0 = 0 \in \mathbb{R}^N$ and $v_1 = 1 \in \mathbb{R}^N$. The numerical results are presented in Table 3, which shows the SNR values for each algorithm, and Figure 4 shows the original, blurred and recovered images.

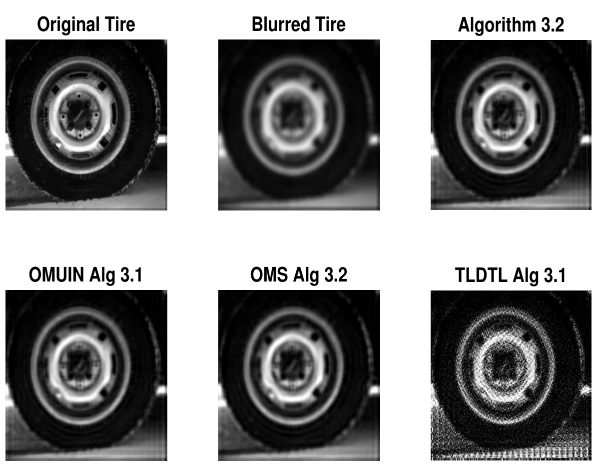

FIGURE 4. Comparison of restored images via various algorithms when the number of iterations is 2500 of Tire image.

TABLE 3. Numerical comparison for Algorithm 3.2, OMUIN Alg 3.1, OMS Alg 3.2 and TLDTL Alg 3.1.

Images	n	Algorithm 3.2	OMUIN Alg 3.1	OMS Alg 3.2	TLDTL Alg 3.1
Tire		SNR	SNR	SNR	SNR
Size=291 × 240	400	34.5659	31.2750	24.71106	17.6431
	1000	34.6643	31.4661	25.7762	17.5033
	1500	34.6928	31.9709	25.7952	17.8669
	2500	35.6374	31.5995	25.8246	17.3219

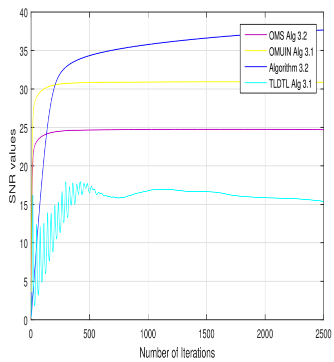

FIGURE 5. Graphs of SNR for Algorithm 3.2, OMUIN Alg 3.1, OMS Alg 3.2 and TLDTL Alg 3.1 of Tire image.

Remark 5.1. *Since the higher the SRN value, the better the quality of recovered image, it evident from Table 3 and Figure 5 that our algorithm 3.2 is more efficient than OMUIN Alg 3.1, OMS Alg 3.2 and TLDTL Alg 3.1.*

6. CONCLUSION

This paper presents a unified iterative method that integrates the subgradient extragradient approach with the golden ratio technique to solve quasimonotone variational inequalities in Hilbert spaces. The proposed algorithm features a dynamic stepsize rule using both linesearch and self-adaptive strategies, eliminating the need for prior knowledge of the Lipschitz constant. Theoretical results establish weak and linear convergence under mild conditions, with linear convergence achieved when the operator is strongly pseudomonotone. Numerical experiments demonstrate that the method outperforms existing algorithms in both speed and efficiency across different problem settings. Additionally, the algorithm is successfully applied to an image restoration problem, where it delivers higher signal-to-noise ratio (SNR) values than comparable methods, confirming its practical utility. By addressing limitations in previous methods and extending their applicability to quasimonotone settings, this work offers a robust and efficient framework for solving a broad class of variational inequality problems

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[1] Department of Mathematics, Sol Plaatje University, Private Bag X5008, Kimberley, 8300, South Africa
Email address: hammedabass548@gmail.com

[2] Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria, South Africa
Email address: ofemaustine@gmail.com

[3] Department of Mathematics and Applied Mathematics, Sefako Makgato Health Science University, P.O. Box 94, Pretoria 0204, South Africa
Email address: maggie.aphane@smu.ac.za