CARPATHIAN J. MATH.
Volume **42** (2026), No. 4,
Pages 871-879

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2026.04.09

An analogue of Tikhonov theorem for singularly perturbed inertial neural networks

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ABSTRACT. The paper investigates inertial neural networks with singularities. Singularly perturbed inertial neural networks represent a class of dynamic systems where the interaction of neurons is considered with perturbations. In this study, the features of stability and behavior of neurons under the influence of small yet significant perturbations, which can substantially affect the network dynamics are considered. An analogue of Tikhonov theorem for the singularly perturbed inertial neural network is formulated and proved. The results can be applied to improve the stability and robustness of neural networks in real-world applications where noise and external influences are present. Numerical example with simulations is given to support the theoretical results.

1. INTRODUCTION

An inertial neural network is a special type of neural network where the behavior of each neuron is modeled based on its inertia, similar to physical systems where changes do not occur instantly, but depend on the previous state and the rate of change. Such networks are often used to model systems with time delay or dynamic changes. In traditional neural networks, the output of a neuron depends only on the input signal and the activation function. In an inertial network, the output of a neuron is determined not only by the current input, but also by its dynamics over time - how quickly states change. This allows neurons to model behavior that does not change instantly, but gradually. An important characteristic of inertial neurons is their ability to "resist" rapid state changes. Similar to a physical system with mass, a neuron takes time to change its state. This inertia means that the behavior of a neuron is influenced as by both current signals and the rate of changes occurring earlier. Thus, inertial neural networks are a powerful tool for modeling systems with time dynamics, taking into account input signals as well as changes in time.

Traditional neural networks operate by propagating information layer by layer without explicitly incorporating temporal or inertial dynamics. However, many real-world phenomena involve dynamics influenced by inertia - whether in physical systems, time-dependent processes, or optimization landscapes. Inertial neural networks aim to bridge this gap by embedding principles of inertia into their architecture, enhancing their capability to model complex dynamic systems and solve problems in physics, control systems, and etc.

The concept of embedding inertial dynamics into neural networks has its origins in studies of optimization methods and physical processes. One of the foundational work on inertial neural networks belongs to Babcock K. et al. [1], who introduced the simple electronic neural networks with added inertia. Alimi A. [2], Lakshmanan S. [3], Zhang Z.Q. [4], formalized this connection by deriving continuous-time models of neural networks

Received: 05.11.2025. In revised form: 12.02.2026. Accepted: 14.04.2026
2020 *Mathematics Subject Classification.* 34D15, 34E15, 92B20.
Key words and phrases. *Inertial neural networks, small parameter, singular perturbation, Tikhonov theorem.*
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using differential equations. These efforts laid the groundwork for integrating inertial dynamics into learning algorithms.

In [5], an inertial neural networks was firstly considered as the following differential equation

$$x_i''(t) = -a_i x_i'(t) - b_i x_i(t) + \sum_{j=1}^{p} c_{ij} f_j(x_j(t)) + v_i(t). \tag{1.1}$$

Many researchers [6, 7, 8, 9, 10, 11, 12], devoted their study to the existence of periodic, almost periodic, unpredictable and stable solutions of the system (1.1).

The concept of inertia in a neural network, which reduces the system's sensitivity to small perturbations, can lead to improved performance in optimization tasks, as it smooths the learning dynamics. The continuous-time formulation is also highly relevant in domains like control theory, signal processing, and robotics, where systems naturally evolve over time and require adaptive mechanisms to learn optimal behaviors in dynamic environments. Singular perturbation theory is a mathematical framework used to study problems that involve multiple timescales, where one timescale is much smaller (or larger) than the other. In the context of differential equations, singular perturbations arise when the system contains a small parameter ε that causes the system to exhibit significantly different behaviors on different timescales. This theory is often used to approximate complex systems by reducing them to simpler subsystems, making it a powerful tool in the analysis of dynamical systems, control systems, and neural networks. Despite the fact that problems with singular perturbations have been known since the beginning of the 20th century, one of the important results obtained in this direction is Tikhonov theorem for singularly perturbed nonlinear differential equations [13, 14].

In this paper, we focus on the following singularly perturbed inertial neural networks

$$\varepsilon x_i''(t) = -a_i x_i'(t) - b_i x_i(t) + \sum_{j=1}^{k} c_{ij} f_j(x_j(t)) + v_i(t), \tag{1.2}$$

with initial conditions: $x_i(0,\varepsilon) = x_i^0, x_i'(0,\varepsilon) = y_i^0$, where $t, x_i : \mathcal{R} \to \mathcal{R}$, $i = 1, ..., k$, k-number of neurons; $x_i(t)$ denotes the state variable of the ith neuron at time t; the second derivative of $x_i(t)$ is called an inertial term of system (1.2); $a_i > 0$ is the damping coefficient; $b_i > 0$ denotes the rate at which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external input; c_{ij}-is constant and denotes the connection strength of the jth neuron on the ith neuron; ε is a positive small parameter; f_i denotes the measures of activation; v_i is the external input on the i-th neuron.

2. Main result

For the system (1.2), we use the following change of variables

$$y_i(t) = x_i'(t). \tag{2.3}$$

By substitution (2.3), inertial neural networks (1.2) can be rewritten as

$$\begin{cases} \varepsilon \frac{dy_i(t)}{dt} = -a_i y_i(t) - b_i x_i(t) + \sum_{j=1}^{k} c_{ij} f_j(x_j(t)) + v_i(t), \\ \frac{dx_i(t)}{dt} = y_i(t), \end{cases} \tag{2.4}$$

with the initial conditions

$$y_i(0,\varepsilon) = y_i^0, x_i(0,\varepsilon) = x_i^0. \tag{2.5}$$

If $\varepsilon = 0$ then we get degenerate system

$$(2.6)\qquad \begin{cases} 0 = -a_i\overline{y}_i(t) - b_i\overline{x}_i(t) + \sum_{j=1}^{k} c_{ij} f_j(\overline{x}_j(t)) + v_i(t), \\ \frac{d\overline{x}_i(t)}{dt} = \overline{y}_i(t), \end{cases}$$

with the initial conditions $x_i(0) = x_i^0$.

Let us assume that the first equation of the system (2.6) has a root $\overline{y}_i = \varphi(\overline{x}_i, t)$, i.e. in our case it has the following form

$$(2.7)\qquad \overline{y}_i(t) = \varphi(\overline{x}_i(t), t) = -\frac{b_i}{a_i}\overline{x}_i(t) + \frac{1}{a_i}\sum_{j=1}^{k} c_{ij} f_j(\overline{x}_j(t)) + \frac{1}{a_i} v_i(t).$$

Applying equality (2.7) we can rewrite the system (2.6) as

$$(2.8)\qquad \frac{d\overline{x}_i(t)}{dt} = -\frac{b_i}{a_i}\overline{x}_i(t) + \frac{1}{a_i}\sum_{j=1}^{k} c_{ij} f_j(\overline{x}_j(t)) + \frac{1}{a_i} v_i(t),$$

with the initial conditions $\overline{x}_i(0) = x_i^0$.

For convenience, let us denote

$$F(y_i, x_i, t) = -a_i y_i(t) - b_i x_i(t) + \sum_{j=1}^{p} c_{ij} f_j(x_j(t)) + v_i(t).$$

The following conditions are required:

C1. The functions $F(y_i, x_i, t), y_i(t)$ are continuous and satisfy the Lipschitz condition with respect to y_i, x_i in some open domain $G = ((y_i, x_i, t) \in \overline{D} = 0 \le t \le T$, $|y_i| < \alpha_i$, $|x_i| \le \beta_i)$, of the space of variables (y_i, x_i, t).

C2. The equation $F(y_i, x_i, t) = 0$ with respect to y_i has a root $y_i(t) = \varphi(x_i(t), t) = -\frac{b_i}{a_i}x_i(t) + \frac{1}{a_i}\sum_{j=1}^{k} c_{ij} f_j(x_j(t)) + \frac{1}{a_i}v_i(t)$ in some limited domain $\overline{D}$ of the space of variables (x_i, t) such that:
(1) the function $\varphi(x_i, t)$ is a continuous function in $\overline{D}$;
(2) the points $\varphi(x_i, t) \in G$ at $(x_i, t) \in \overline{D}$;
(3) the root $y_i(t)$ is isolated in $\overline{D}$, i.e, there exist $\eta > 0$ such that $F(y_i, x_i, t) \neq 0$ for $0 < |y_i(t) + \frac{b_i}{a_i}x_i(t) - \frac{1}{a_i}\sum_{j=1}^{k} c_{ij} f_j(x_j(t)) - \frac{1}{a_i}v_i(t)| < \eta$, $(x_i, t) \in \overline{D}$.

C3. The system (2.8) has a unique solution $\overline{x}_i(t)$ on the segment $0 \le t \le T$, and the points $(\overline{x}_i(t), t) \in D$ as $t \in [0, T]$, where D is the set of interior points of the domain $\overline{D}$. In addition, assume that $\overline{y}_i(t)$ satisfy the Lipschitz condition with respect to x_i in $\overline{D}$.

Let us introduce an adjoint system in the form

$$(2.9)\qquad \frac{d\overline{y}_i}{d\tau} = -a_i\overline{y}_i(\tau\varepsilon) - b_i x_i(\tau\varepsilon) + \sum_{j=1}^{k} c_{ij} f_j(x_j(\tau\varepsilon)) + v_i(\tau\varepsilon), \quad \tau > 0,$$

in which $\tau = \frac{t}{\varepsilon}$ and x_i, t are considered as parameters. By virtue of C2, $\overline{y}_i(t) = \varphi(x_i(t), t) = -\frac{b_i}{a_i}x_i(t) + \frac{1}{a_i}\sum_{j=1}^{k} c_{ij} f_j(x_j(t)) + \frac{1}{a_i}v_i(t)$ is an isolated equilibrium point of the adjoint system at $(x_i, t) \in \overline{D}$.

C4. The equilibrium point $\overline{y}_i = \varphi(x_i, t)$ of system (2.9) is asymptotically stable uniformly with respect to $(x_i, t) \in \overline{D}$.

The last condition means, that $\forall\mu > 0, \exists\overline{\delta}(\mu)$ (common to all $(x_i, t) \in \overline{D}$) such that, if $|y_i(0)-\varphi(x_i,t)| < \overline{\delta}(\mu)$ then condition $|y_i(\tau)-\varphi(x_i,t)| < \mu$ is true. Fulfillment of condition C4 means that the root $y_i = \varphi(x_i, t)$ is stable in domain $\overline{D}$.

Let us consider the adjoint system (2.9), when $t = 0$

$$\frac{d\overline{y}_i}{d\tau} = -a_i\overline{y}_i - b_i x_i^0 + \sum_{j=1}^{k} c_{ij} f_j(x_j^0) + v_i(0), \tag{2.10}$$

with initial conditions

$$\overline{y}_i(0) = y_i^0. \tag{2.11}$$

Since, the initial value of y_i^0 is, generally, not close to the equilibrium point of $\varphi(x_i^0, 0) = -\frac{b_i}{a_i}x_i^0 + \frac{1}{a_i}\sum_{j=1}^{k} c_{ij}f_j(x_j^0) + \frac{1}{a_i}v_i(0)$, the solution of (2.10)-(2.11) problem may not to tend to $\varphi(x_i^0, 0)$ as $\tau \to \infty$.

C5. Let the solution of problem (2.10), (2.11) satisfy the conditions:

(1) $\overline{y}_i(\tau) \to -\frac{b_i}{a_i}x_i^0 + \frac{1}{a_i}\sum_{j=1}^{k} c_{ij}f_j(x_j^0) + \frac{1}{a_i}v_i(0)$ at $\tau \to \infty$;

(2) the points $(\overline{y}_i(\tau), x_i^0, 0) \in G$ at $\tau \geq 0$.

In this case, the initial value y_i^0 belongs to the domain of attraction of the equilibrium point $(\overline{y}_i(\tau), x_i^0, 0)$.

The main result of the paper is the following Tikhonov-type theorem for singularly perturbed inertial neural networks (2.4).

Theorem 2.1. *Suppose that conditions C1-C5 are satisfied. Then, for sufficiently small ε, solutions $y_i(t,\varepsilon)$ and $x_i(t,\varepsilon)$ of problem (2.4) with initial conditions (2.5) exist on $0 \leq t \leq T$, are unique, and satisfy the equalities*

$$\lim_{\varepsilon\to 0} x_i(t,\varepsilon) = \overline{x}_i(t), 0 \leq t \leq T, \tag{2.12}$$

$$\lim_{\varepsilon\to 0} y_i(t,\varepsilon) = \overline{y}_i(t) = \varphi(\overline{x}_i, t), 0 < t \leq T. \tag{2.13}$$

Let us denote by U_μ the set of points in space (y_i, x_i, t) such that

$$U_\mu = (y_i, x_i, t) : |y_i - \varphi(x_i, t)| < \mu,\ (x_i, t) \in D. \tag{2.14}$$

Then for the closure of the set U_μ, we have the following set

$$\overline{U}_\mu = (y_i, x_i, t) : |y_i - \varphi(x_i, t)| \leq \mu,\ (x_i, t) \in \overline{D}. \tag{2.15}$$

To prove the theorem we need the following auxiliary lemma, the proof of which can be found in [15].

Lemma 2.1. *Suppose that conditions C1-C4 are hold, and let $\mu > 0$ be an arbitrary small number such that $\overline{U}_\mu \subset G$. Then there exist a numbers $\delta = \delta(\mu)$ and μ_0 such that for $0 < \varepsilon \leq \mu_0$ the solution $y_i(t,\varepsilon), x_i(t,\epsilon)$ of system (2.4), the initial point of which $(y(t_0,\varepsilon), x_i(t_0,\varepsilon)) \in U_\delta$, exists, unique and does not leave U_μ for $t > t_0$ until $(x_i(t,\varepsilon), t) \in D$.*

Proof. 2.1 Let us fix an arbitrary $\mu > 0$, such that $\overline{U}_\mu \subset G$. We take $\delta = \delta(\mu)$ defined in Lemma 2.1, and consider the adjoint system (2.10)-(2.11). Condition C5 implies that there exists a number $\tau_0 = \tau_0(\delta) > 0$, such that

$$|\overline{y}_i(\tau_0) - \varphi(x_i^0, 0)| < \frac{1}{3}\delta. \tag{2.16}$$

Let's replace $t = \tau\mu$ in the problem (2.4)-(2.5). We get

$$\begin{cases} \frac{dy_i}{d\tau} = -a_i y_i(\tau\varepsilon) - b_i x_i(\tau\varepsilon) + \sum_{j=1}^{k} c_{ij} f_j(x_j(\tau\varepsilon)) + v_i(\tau\varepsilon), \quad y_i\big|_{\tau=0} = y_i^0, \\ \frac{dx_i}{d\tau} = \varepsilon y_i(\tau\varepsilon), \quad x_i\big|_{\tau=0} = x_i^0. \end{cases} \tag{2.17}$$

Since, by virtue of condition C5, the points $(\overline{y}(\tau), x_i^0, 0) \in G$ for $\tau \geq 0$, then according to the theorem on the solution of a regularly perturbed problem (Theorem 2.2, [15]) for sufficiently small ε, the solution $y_i(\tau\varepsilon, \varepsilon)$, $x_i(\tau\varepsilon, \varepsilon)$ of problem (2.17) exist on the segment $0 \leq \tau \leq \tau_0$, is unique and uniform with respect to τ, and the limit equalities

$$\lim_{\varepsilon \to 0} y_i(\tau\varepsilon, \varepsilon) = \overline{y}_i(\tau),$$

$$\lim_{\varepsilon \to 0} x_i(\tau\varepsilon, \varepsilon) = x_i^0, \tag{2.18}$$

are satisfied. Therefore, there exist $\varepsilon_0 > 0$, such that if $0 < \varepsilon \leq \varepsilon_0$ then the inequality

$$|y_i(\tau\varepsilon, \varepsilon) - \overline{y}_i(\tau)| < \frac{1}{3}\delta, \tag{2.19}$$

will be satisfied for $0 \leq \tau \leq \tau_0$.

Let us take sufficiently small ε such that $(x_i(\tau\varepsilon, \varepsilon), \tau\varepsilon) \in D$, and

$$|\varphi(x_i(\tau_0\varepsilon, \varepsilon), \tau_0\varepsilon) - \varphi(x_i^0, 0)| < \frac{1}{3}\delta, \tag{2.20}$$

for $0 \leq \tau \leq \tau_0$, $0 < \varepsilon \leq \varepsilon_0$, which is possible due to the limitations of functions x_i, y_i. Then the inequalities (2.16), (2.19) and (2.20) give that

$$|y_i(\tau_0\varepsilon, \varepsilon) - \varphi(x_i(\tau_0\varepsilon, \varepsilon), \tau_0\varepsilon)| < \delta. \tag{2.21}$$

Thus, for $t \leq t_0 = \tau_0\varepsilon$ the solutions $y_i(t, \varepsilon) = y_i(\tau\varepsilon, \varepsilon), x_i(t, \varepsilon) = x_i(\tau\varepsilon, \varepsilon)$ of problem (2.4)-(2.5) satisfies the condition

$$(y_i(t_0, \varepsilon), x_i(t_0, \varepsilon)) \in U_\delta. \tag{2.22}$$

By Lemma 2.1, the solution $y_i(t, \varepsilon), x_i(t, \varepsilon)$ is uniquely extendable for $t > t_0$ and does not leave U_μ until $(x_i(t, \varepsilon), t) \in D$. Hence, for $t \geq t_0$ as long as $(x_i(t, \varepsilon), t) \in D$, the following equality holds

$$y_i(t, \varepsilon) = -\frac{b_i}{a_i} x_i(t) + \frac{1}{a_i} \sum_{j=1}^{k} c_{ij} f_j(x_j(t)) + \frac{1}{a_i} v_i(t) + \gamma_i(t, \varepsilon),$$

where $\gamma_i(t, \varepsilon)$ is some continuous function satisfying the inequality $||\gamma_i(t, \varepsilon)|| < \mu$, and $x_i(t, \varepsilon)$ is the solution of the problem

$$\begin{cases} \frac{dx_i}{dt} = -\frac{b_i}{a_i} x_i(t) + \frac{1}{a_i} \sum_{j=1}^{k} c_{ij} f_j(x_j(t)) + \frac{1}{a_i} v_i(t) + \gamma_i(t, \varepsilon), \\ x_i\big|_{t=t_0} = x_i(t_0, \varepsilon) = x_i^0 + \sigma_i(\varepsilon), \end{cases} \tag{2.23}$$

where $\sigma_i(\varepsilon) \to 0$ at $\varepsilon \to 0$ by virtue of (2.18). Now, let us note the following property of system (2.23). From requirement C3 and theorem of continuous dependence on parameters (Theorem 2.2, [15]) it follows that for any $\eta > 0$ there exists an $\mu_0(\eta)$ such that: if $\gamma_i(t, \varepsilon)$ is defined and continuous at $t_0 < T_0 \leq T$ and satisfies the inequality $|\gamma_i(t, \varepsilon)| \leq \mu_0(\eta)$, $t_0 \leq t \leq T$, and $|\sigma_i(\varepsilon)| \leq \mu_0(\eta)$ for $t_0 \leq \mu_0(\eta)$ then the solution $x_i(t, \varepsilon)$ of problem (2.23) exists on the segment $t_0 \leq t \leq T_0$, points $(x_i(t, \varepsilon), t) \in D$ at $t \in [t_0, T]$ and the inequalities $|x_i(t, \varepsilon) - \overline{x}_i(t)| \leq \eta$ is valid at $t_0 \leq t \leq T_0$. Let us take an arbitrary small numbers $\eta > 0$, μ from the interval $0 < \mu \leq \mu_0(\eta)$, and ε_0, $0 < \varepsilon \leq \varepsilon_0$ such that condition (2.22) and the inequalities

$$|x_i(t, \varepsilon) - x_i^0| \leq \frac{\eta}{2}, 0 \leq t \leq t_0, \tag{2.24}$$

$$|\overline{x}_i(t) - x_i^0| \leq \frac{\eta}{2}, 0 \leq t \leq t_0 \tag{2.25}$$

are fulfilled.

From inequalities (2.24) and (2.25) it follows that

$$|x_i(t,\varepsilon) - \overline{x}_i(t)| \le \eta, \tag{2.26}$$

for $0 \le t \le t_0, 0 < \varepsilon \le \varepsilon_0$.

It is also true that the solutions $y_i(t,\varepsilon), x_i(t,\varepsilon)$ of problem (2.4)-(2.5) located in U_δ if $t = t_0$. They are unique continued up to $t = T$ and following inequalities

$$|x_i(t,\varepsilon) - \overline{x}_i(t)| \le \eta, t_0 \le t \le T,$$
$$|y_i(t,\varepsilon) + \frac{b_i}{a_i}x_i(t) - \frac{1}{a_i}\sum_{j=1}^{k} c_{ij}f_j(x_j(t)) - \frac{1}{a_i}v_i(t) - \gamma(t,\varepsilon)| < \mu,$$

are hold for $t_0 \le t \le T$. Thus, we obtained the unique continuable solutions $y_i(t,\varepsilon), x_i(t,\varepsilon)$ for $0 \le t \le T$. The theorem is proved. □

3. Numerical example

Let us consider the following singularly perturbed inertial neural network

$$\varepsilon\frac{d^2x_i(t)}{dt^2} = -a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{3} c_{ij}f_j(x_j(t)) + v_i(t) \tag{3.27}$$

where $\varepsilon > 0$ is a small parameter, $i = 1,2,3$, $a_1 = 2$, $a_2 = 0.5$, $a_3 = 3$, $b_1 = 1$, $b_2 = 3$, $b_3 = 2$, $f_1(x) = \frac{1}{1+\exp(-x)}$, $f_2(x) = \arctan x$, $f_3(x) = \tanh x$,

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 0.02 & 0.01 & 0.03 \\ 0.04 & 0.03 & 0.05 \\ 0.03 & 0.01 & 0.06 \end{pmatrix},$$

and $v_1 = \exp(-t)\sin t, v_2 = \frac{1}{1+t^2}, v_3 = \arctan t$. The system (3.27) can be rewritten as

$$\begin{cases} \frac{dx_1(t)}{dt} = y_1(t), \\ \frac{dx_2(t)}{dt} = y_2(t), \\ \frac{dx_3(t)}{dt} = y_3(t), \\ \varepsilon\frac{dy_1(t)}{dt} = -2y_1(t) - x_1(t) + \frac{0.02}{1+\exp(-x_1(t))} + 0.01\arctan x_2(t) \\ +0.03\tanh x_3(t) + \exp(-t)\sin t, \\ \varepsilon\frac{dy_2(t)}{dt} = -0.5y_2(t) - 3x_2(t) + \frac{0.04}{1+\exp(-x_1(t))} + 0.03\arctan x_2(t) \\ +0.05\tanh x_3(t) + \frac{1}{1+t^2}, \\ \varepsilon\frac{dy_3(t)}{dt} = -3y_3(t) - 2x_3(t) + \frac{0.03}{1+\exp(-x_1(t))} + 0.01\arctan x_2(t) \\ +0.06\tanh x_3(t) + \arctan t. \end{cases} \tag{3.28}$$

Consider neural network (3.27), with initial conditions $x_1(0,\varepsilon) = 2.5$, $x_2(0,\varepsilon) = 1$, $x_3(0,\varepsilon) = 1$, $y_1(0,\varepsilon) = 1$, $y_2(0,\varepsilon) = 2$, $y_3(0,\varepsilon) = 1.5$. When $\varepsilon = 0$ we obtain the degenerate system

$$(3.29)\quad \begin{cases} \frac{d\overline{x}_1(t)}{dt} = \overline{y}_1(t), \\ \frac{d\overline{x}_2(t)}{dt} = \overline{y}_2(t), \\ \frac{d\overline{x}_3(t)}{dt} = \overline{y}_3(t), \\ 0 = -2\overline{y}_1(t) - \overline{x}_1(t) + \frac{0.02}{1+\exp(-\overline{x}_1(t))} + 0.01\arctan\overline{x}_2(t) \\ +0.03\tanh\overline{x}_3(t) + \exp(-t)\sin t, \\ 0 = -0.5\overline{y}_2(t) - 3\overline{x}_2(t) + \frac{0.04}{1+\exp(-\overline{x}_1(t))} + 0.03\arctan\overline{x}_2(t) \\ +0.05\tanh\overline{x}_3(t) + \frac{1}{1+t^2}, \\ 0 = -3\overline{y}_3(t) - 2\overline{x}_3(t) + \frac{0.03}{1+\exp(-\overline{x}_1(t))} + 0.01\arctan\overline{x}_2(t) \\ +0.06\tanh\overline{x}_3(t) + \arctan t, \end{cases}$$

with initial conditions $\overline{x}_1(0) = 2.5$, $\overline{x}_2(0) = 1$, $\overline{x}_3(0) = 1$.

All coefficients of system (3.28) are satisfy to the assumptions of the Theorem 2.1. Figure 1 shows the trajectories $x_i(t), i = 1, 2, 3$, of the system (3.28) at various values of ε. It can be seen that all components of $x_i(t)$ change smoothly over the interval $t \in [0, 10]$ and with decreasing ε, the graphs converge without sudden initial jumps. This is consistent with the model structure: there is no small parameter in the equations for x, so the dynamics of the x variables is regular and demonstrates smooth convergence to the solution of the reduced system. Figure 2 presents the solutions $y_i(t), i = 1, 2, 3$, for the same values of ε. Near $t = 0$, a rapid transient is observed: the trajectories undergo a sharp adjustment and only afterwards enter a slow evolution regime. As ε decreases, this fast segment becomes increasingly localized in time, indicating the presence of a boundary layer, which is typical for singularly perturbed equations where the small parameter generates a fast time scale. Hence, the variables $x_i(t)$ display a regular evolution and a smooth dependence on ε without boundary-layer effects, whereas the variables $y_i(t)$ exhibit a pronounced two-time-scale behavior: a fast initial layer followed by slow motion. The convergence of $x_i(t)$ to the reduced regime is essentially uniform, while for $y_i(t)$ it is non-uniform due to the formation of a boundary layer near the initial time.

4. Conclusions

In this paper, an analogue of Tikhonov's theorem is considered for a singularly perturbed inertial neural network modeled by a system of ordinary differential equations with a small parameter at the highest derivative. A qualitative analysis of the asymptotic behavior of the solutions as the small parameter tends to zero is carried out. It is shown that, under certain conditions on the activation functions and network parameters, the solution of the singularly perturbed system converges to the solution of the corresponding reduced system describing the slow dynamics. Thus, an analogue of the classical Tikhonov theorem is established for the considered class of neural models.

The obtained results may serve as a theoretical basis for simplifying the analysis of the dynamics of inertial neural networks, especially in modeling cognitive processes and information processing in biological and technical systems. Future work will aim to extend the analysis to more general classes of networks, including those with random connections, delays, and adaptive parameters.

Acknowledgments

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23487275).

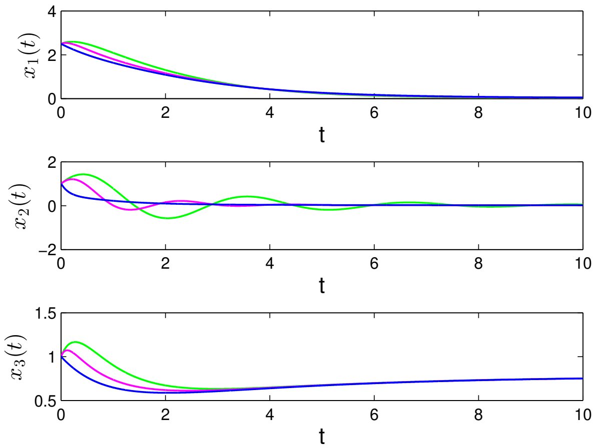

FIGURE 1. Solutions $x_i, i = 1, 2, 3$ of the system (3.28) with initial values $x_1(0, \varepsilon) = 2.5$, $x_{2,\varepsilon} = 1$, $x_3(0, \varepsilon) = 1$, where green, magenta and blue lines are graphs of the solutions for $\varepsilon : 0.7, 0.3, 0.001$, respectively.

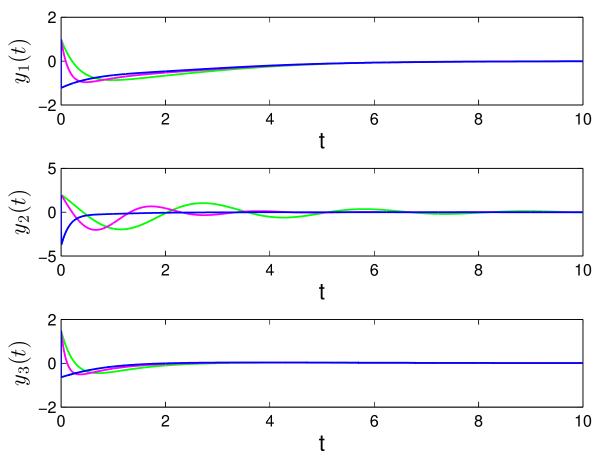

FIGURE 2. The green, magenta and blue lines are graphs of solutions $y_i(t)$ of the system (3.28) with initial value (1, 2, 1.5) for $\varepsilon : 0.7, 0.3, 0.001$, respectively.

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