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Regularity and maximal subsemigroups in semigroups of linear transformations whose restrictions belong to a general linear group

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ABSTRACT. Let V be a vector space over a field $\mathbb{F}$ and U a fixed subspace of V. Define the set $L_{GL(U)}(V)$ of all linear transformations on V whose restrictions to U are elements of the general linear group $GL(U)$. Then $L_{GL(U)}(V)$ is a regular subsemigroup of the full linear transformation semigroup $L(V)$ under composition. In this paper, we characterize left (right) regular elements in $L_{GL(U)}(V)$, and determine when $L_{GL(U)}(V)$ is a left (right) regular semigroup. We also provide a complete description of unit regular elements in $L_{GL(U)}(V)$ and determine when $L_{GL(U)}(V)$ is a unit regular semigroup. Moreover, the number of these elements is computed when V is finite-dimensional and $\mathbb{F}$ is a finite field. Finally, we classify all maximal subsemigroups of $L_{GL(U)}(V)$ and describe its maximal regular subsemigroups.

1. INTRODUCTION

Consider $T(X)$, the full transformation semigroup on a set X under composition, and define $S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$ for a nonempty subset $Y \subseteq X$. It follows that $S(X,Y)$ forms a subsemigroup of $T(X)$. Given a fixed nonempty subset Y of X, let $G(Y)$ denote the symmetric group on Y. In 2016, E. Laysirikul [7] introduced

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$$

where $\alpha|_Y$ is the restriction of α to Y. The author established that $PG_Y(X)$ is a subsemigroup of $S(X,Y)$. Several connections between $PG_Y(X)$, its subsemigroups, and $S(X,Y)$ were investigated. Furthermore, the author proved that $PG_Y(X)$ is a regular semigroup and provided characterizations for left regular, right regular, and completely regular elements in $PG_Y(X)$. In 2021, W. Sommanee [12] provided a simplified characterization of Green's relations on $PG_Y(X)$ and determined the structure of its ideals. The author also established several isomorphism theorems concerning $PG_Y(X)$. In the case where X is finite, the author determined the cardinality of $PG_Y(X)$, enumerated its idempotent elements, and computed the rank of $PG_Y(X)$.

Recently, Y. Chaiya [1] characterized all intra-regular and unit regular elements within $PG_Y(X)$ and provided necessary and sufficient conditions for $PG_Y(X)$ to possess these regularity properties. When X is finite, enumeration formulas for the number of such elements were derived. Additionally, the author provided a complete classification of the maximal subsemigroups of $PG_Y(X)$ and demonstrated that these maximal subsemigroups coincide with the maximal regular subsemigroups of $PG_Y(X)$ under the condition that $X \setminus Y$ is finite.

Given any subset Y of X and any subsemigroup $\mathbb{S}(Y)$ of $T(Y)$, we denote by $T_{\mathbb{S}(Y)}(X)$ the semigroup consisting of all transformations $\alpha \in T(X)$ satisfying $\alpha|_Y \in \mathbb{S}(Y)$. It is clear that this semigroup generalizes both $S(X,Y)$ and $PG_Y(X)$ since $S(X,Y) = T_{T(Y)}(X)$ and

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$PG_Y(X) = T_{G(Y)}(X)$. In 2022, J. Konieczny [6] characterized the regular elements of $T_{\mathbb{S}(Y)}(X)$ and established necessary and sufficient conditions for $T_{\mathbb{S}(Y)}(X)$ to be a regular semigroup, an inverse semigroup, or a completely regular semigroup. Under the assumption that $\mathbb{S}(Y)$ contains the identity transformation id_Y, the author characterized Green's relations on $T_{\mathbb{S}(Y)}(X)$ using the corresponding Green's relations on $\mathbb{S}(Y)$.

Similarly to $S(X,Y)$, $PG_Y(X)$, and $T_{\mathbb{S}(Y)}(X)$, linear analogues of these semigroups have been investigated as follows.

For a vector space V, let $L(V)$ be the semigroup (under composition) of all linear transformations from V into itself. It is well-known that $L(V)$ is a regular semigroup. Given a fixed subspace $W \subseteq V$, P. Honyam and J. Sanwong [4] introduced $S(V,W)$ as the subsemigroup of $L(V)$ consisting of all linear transformations that leave a subspace W invariant. The authors described the ideals and Green's relations within $S(V,W)$. Additionally, they demonstrated that provided W is a non-trivial, proper subspace of V, $S(V,W)$ is never isomorphic to $L(U)$ for any arbitrary vector space U.

Let V be a vector space over a field $\mathbb{F}$ and $L(V)$ the full linear transformation semigroup on V under composition. For a subspace W of V and a subsemigroup $\mathbb{S}(W)$ of $L(W)$, M. Sarkar and S. N. Singh [11] introduced an analogous subsemigroup $L_{\mathbb{S}(W)}(V) = \{\alpha \in L(V) : \alpha|_W \in \mathbb{S}(W)\}$ of $L(V)$. The authors characterized the regular elements of $L_{\mathbb{S}(W)}(V)$ and established conditions under which $L_{\mathbb{S}(W)}(V)$ is a regular semigroup, an inverse semigroup, or a completely regular semigroup. When $\mathbb{S}(Y)$ (respectively, $\mathbb{S}(W)$) contains the identity element of $T(Y)$ (respectively, $L(W)$), the authors characterized unit regular elements in $T_{\mathbb{S}(Y)}(X)$ (respectively, $L_{\mathbb{S}(W)}(V)$) and provided necessary and sufficient conditions for $T_{\mathbb{S}(Y)}(X)$ (respectively, $L_{\mathbb{S}(W)}(V)$) to be a unit regular semigroup.

Let V denote a vector space with U as a fixed subspace. Recently, K. Sangkhanan [10] investigated the semigroup $L_{GL(U)}(V) = \{\alpha \in L(V) : \alpha|_U \in GL(U)\}$, consisting of all linear transformations on V whose restrictions to U belong to the general linear group $GL(U)$, as a subsemigroup of $L(V)$. This work provides a complete description of Green's relations and the ideal structure of $L_{GL(U)}(V)$, and identifies both the minimal ideal and the set of all minimal idempotents. When V is a finite-dimensional vector space over a finite field, the author established an isomorphism theorem for $L_{GL(U)}(V)$. Additionally, the paper determined a generating set for this semigroup.

We observe that if $U = V$, then the set $L_{GL(U)}(V)$ is in fact equal to $GL(U)$, the general linear group of U. In the particular case when $U = \{0\}$, the set $L_{GL(U)}(V)$ coincides with the space of all linear maps on V, that is, $L_{GL(U)}(V) = L(V)$. Moreover, since $L_{GL(U)}(V)$ contains the identity map id_V as its identity element, we can write $L_{GL(U)}(V)^1 = L_{GL(U)}(V)$.

The present study is based on recent work on transformation semigroups, including [1, 6, 7, 11, 12]. Even with these results, some basic questions about $L_{GL(U)}(V)$ are still open. In [10], Green's relations, ideals, and generating sets of $L_{GL(U)}(V)$ were studied. However, left regular, right regular, intra-regular, and unit regular elements were not fully described, and maximal subsemigroups were not classified. These are central topics in semigroup theory.

In this paper, we fill this gap. We give complete conditions for these regularity properties and classify maximal and maximal regular subsemigroups of $L_{GL(U)}(V)$. So, $L_{GL(U)}(V)$ becomes a clear linear analogue of $PG_Y(X)$ in this direction.

We now show how our results fit with earlier work. For sets, the paper [1] studied regularity and maximal subsemigroups of $PG_Y(X)$. For vector spaces, the paper [11] studied regularity of $L_{\mathbb{S}(W)}(V)$. Our semigroup $L_{GL(U)}(V)$ is a special case ($\mathbb{S}(W) = GL(U)$) where we can describe the structure more clearly.

Compared with [11], we give explicit conditions for $L_{GL(U)}(V)$ and fully classify its maximal subsemigroups. Our results are similar to those in $PG_Y(X)$ (from [1]), with $\operatorname{codim}(U)$ playing the same role as $|X \setminus Y|$, but we find different types of maximal subsemigroups.

2. Preliminaries

We commence by introducing the notation and preliminary results from linear algebra and semigroup theory that are employed throughout this paper. For any terminology not defined explicitly here, we refer the reader to [2,3,5,8].

Throughout, we let U be a subspace of a vector space V, and we use $\langle e_i \rangle = \langle \{e_i : i \in I\} \rangle$ to denote the subspace spanned by a linearly independent set $\{e_i\} = \{e_i : i \in I\}$ in V. When we write $U = \langle e_i \rangle$, we mean that $\{e_i\}$ forms a basis for U, which implies that the dimension of U, written $\dim U$, is equal to $|I|$. It is well known that $\dim U \leq \dim V$, and furthermore, if $\dim U = \dim V$ when V is finite-dimensional, then $U = V$.

For each linear transformation $\alpha \in L(V)$, we write $\ker \alpha$ and $V\alpha$ for its *kernel* and *image*, respectively. The *rank* of α, denoted $\operatorname{rank}(\alpha)$, is defined as the dimension of its image, that is, $\operatorname{rank}(\alpha) = \dim(V\alpha)$. The *nullity* of α, denoted $\operatorname{null}(\alpha)$, is defined as the dimension of its kernel, that is, $\operatorname{null}(\alpha) = \dim(\ker \alpha)$.

Consider subspaces M and N of V. The *sum* of M and N, denoted $M+N$, consists of all vectors that can be expressed as the sum of an element from M and an element from N:

$$M + N = \{m + n \mid m \in M, n \in N\}.$$

The sum $M+N$ is the smallest subspace of V that contains both M and N. We say that V is an *internal direct sum* of subspaces S_1 and S_2, denoted $V = S_1 \oplus S_2$, provided that $V = S_1 + S_2$ and $S_1 \cap S_2 = \{0\}$. When V can be expressed as an internal direct sum of subspaces $\{S_1, S_2, \ldots, S_n\}$, we write $V = S_1 \oplus S_2 \oplus \cdots \oplus S_n$. If $V = S \oplus T$, then T is called a *complement* of S in V. Note that although a subspace may have many complements, all such complements are isomorphic to one another. For a finite-dimensional vector space V over a finite field $\mathbb{F}$, the number of distinct complements of a subspace U in V is given by the following theorem.

Theorem 2.1. [13, Theorem 6] *Let U be a k-dimensional subspace of an n-dimensional vector space V over a finite field $\mathbb{F}$. Then there are $|\mathbb{F}|^{k(n-k)}$ distinct complements for U in V.*

Given a subspace U of a vector space V, the *quotient space* V/U is the collection of all cosets of U in V. Every complement of U in V is isomorphic to V/U and hence to each other. The dimension of V/U is termed the *codimension* of U in V and is denoted $\operatorname{codim}(U)$. When V is finite-dimensional, we have $\operatorname{codim}(U) = \dim V - \dim U$.

For a vector space V with a subset $\{u_i\}$, the expression $\sum a_i u_i$ denotes a finite linear combination of the form

$$a_{i_1} u_{i_1} + a_{i_2} u_{i_2} + \cdots + a_{i_n} u_{i_n}$$

where n is a positive integer, $u_{i_1}, u_{i_2}, \ldots, u_{i_n}$ belong to $\{u_i\}$, and $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ are scalars. For $\alpha \in L(V)$ and a subspace U of V, writing $U\alpha = \langle u_j \alpha \rangle$ means that each u_j lies in U and the set $\{u_j\}$ is linearly independent. Moreover, if $V\alpha = \langle v_i \alpha \rangle$, then we can decompose V as $V = \ker \alpha \oplus \langle v_i \rangle$.

Following the notational convention in [3, p. 241], we represent any α in $T(X)$ as

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}.$$

Here, the index i runs over some index set I, and $\{a_i\}$ is shorthand for $\{a_i : i \in I\}$, where $X\alpha = \{a_i\}$ and $a_i \alpha^{-1} = A_i$.

This notation naturally extends to elements of $L(V)$. To define a linear transformation $\alpha \in L(V)$, one chooses a basis $\{e_i\}$ for V and a subset $\{a_i\}$ of V, sets $e_i\alpha = a_i$ for each $i \in I$, and extends this assignment linearly to all of V. Consequently, in a suitable context with specified $\{e_i\}$ and $\{a_i\}$, any $\alpha \in L(V)$ can be compactly written as

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

Our approach depends on the following classical results from linear algebra:

Theorem 2.2. *Let V and W be vector spaces over a field $\mathbb{F}$. Then $V \cong W$ if and only if* $\dim V = \dim W$.

Theorem 2.3. *Let S and T be subspaces of a vector space V satisfying $S \cap T = \{0\}$. Then* $\dim(S \oplus T) = \dim S + \dim T$.

3. Regularity

This section is devoted to studying different notions of regularity in the semigroup $L_{GL(U)}(V)$. We aim to establish explicit algebraic conditions characterizing when an element $\alpha \in L_{GL(U)}(V)$ possesses left regularity, right regularity, intra-regularity, or unit regularity. Furthermore, we derive from these characterizations the necessary and sufficient conditions under which $L_{GL(U)}(V)$ itself exhibits these regularity properties. At the end of the section, we compute the number of such regular elements when V is finite-dimensional and $\mathbb{F}$ is a finite field. We start by reviewing several fundamental definitions from semigroup theory.

Recall that an element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is said to be *regular* if every element is regular. An element b is called an *inverse* of a if

$$aba = a \quad \text{and} \quad bab = b.$$

A semigroup S is called an *inverse semigroup* if each element has a unique inverse.

An element a in a semigroup S is called *left regular* if there exists $x \in S$ such that $a = xa^2$, *right regular* if there exists $x \in S$ such that $a = a^2x$, and *intra-regular* if $a = xa^2y$ for some $x, y \in S$. The element a is said to be *completely regular* if there exists $x \in S$ such that $a = axa$ and $ax = xa$. Every completely regular element is both left and right regular, and it is well known that a is completely regular if and only if it is both left and right regular. If every element of S is left (right, intra-, completely) regular, then S is called a *left (right, intra-, completely) regular semigroup*.

To prove our main results, we require the following characterization of Green's relations on $L_{GL(U)}(V)$ established in [10].

Theorem 3.4. [10, Theorem 3.5] *Let $\alpha, \beta \in L_{GL(U)}(V)$. Then the following statements hold.*

(i) $(\alpha, \beta) \in \mathscr{L}$ *if and only if* $V\alpha = V\beta$.
(ii) $(\alpha, \beta) \in \mathscr{R}$ *if and only if* $\ker \alpha = \ker \beta$.
(iii) $(\alpha, \beta) \in \mathscr{J}$ *if and only if* $\dim(V\alpha/U) = \dim(V\beta/U)$.

We are now in a position to characterize left regular, right regular, and intra-regular elements in $L_{GL(U)}(V)$. It is straightforward to verify that an element a in a semigroup S is left (right) regular if and only if $a\mathscr{L}a^2$ ($a\mathscr{R}a^2$). Therefore, by Theorem 3.4 (i) and (ii), we obtain the following characterizations immediately.

Theorem 3.5. *Let $\alpha \in L_{GL(U)}(V)$. Then α is left regular if and only if $V\alpha = V\alpha^2$.*

Theorem 3.6. *Let $\alpha \in L_{GL(U)}(V)$. Then α is right regular if and only if* $\ker \alpha = \ker \alpha^2$.

Furthermore, if S is a monoid, then $a \in S$ is intra-regular if and only if $a \mathscr{J} a^2$. By Theorem 3.4 (iii), we obtain the following characterization of intra-regular elements in $L_{GL(U)}(V)$.

Theorem 3.7. *Let $\alpha \in L_{GL(U)}(V)$. Then α is intra-regular if and only if* $\dim(V\alpha/U) = \dim(V\alpha^2/U)$.

We are now able to prove the equivalence of left regularity, right regularity, and intra-regularity for elements in $L_{GL(U)}(V)$ with finite codimension.

Theorem 3.8. *Let $\alpha \in L_{GL(U)}(V)$ be such that* $\dim(V\alpha/U)$ *is finite. Then the following statements are equivalent:*

(i) α is left regular; (ii) α is right regular; (iii) α is intra-regular.

Proof. (i) $\Rightarrow$ (ii). Assume that α is left regular. By Theorem 3.5, we have $V\alpha = V\alpha^2$. Let $V\alpha^2 = \langle v_i\alpha^2\rangle \oplus U$. Then $V = \ker\alpha^2 \oplus \langle v_i\rangle \oplus U$. To show that $\{v_i\alpha\}$ is linearly independent, let $\sum a_iv_i\alpha = 0$ for some scalars $a_i \in \mathbb{F}$ for all $i \in I$. Then $\sum a_iv_i\alpha^2 = 0$ which implies that $a_i = 0$ for all $i \in I$ since $\{v_i\alpha^2\}$ is linearly independent. We claim that $\langle v_i\alpha\rangle \cap U = \{0\}$. Indeed, let $v \in \langle v_i\alpha\rangle \cap U$. Then $v = \sum a_iv_i\alpha$ for some scalars $a_i \in \mathbb{F}$ for all $i \in I$. Then $\sum a_iv_i\alpha^2 = v\alpha \in U\alpha = U$ from which it follows that $a_i = 0$ for all $i \in I$. Hence we can write $V\alpha = \langle v_i\alpha\rangle \oplus \langle v_j\alpha\rangle \oplus U$. Since $V\alpha = V\alpha^2$, we obtain $|I| + |J| = \dim(V\alpha/U) = \dim(V\alpha^2/U) = |I|$. Moreover, since $\dim(V\alpha/U)$ is finite, we must have $|J| = 0$. It is concluded that $V\alpha = \langle v_i\alpha\rangle \oplus U$ and so $V = \ker\alpha \oplus \langle v_i\rangle \oplus U$. We can write

$$\alpha = \begin{pmatrix} w_s & v_i & u_k \\ 0 & v_i\alpha & u_k\alpha \end{pmatrix}$$

where $\ker\alpha = \langle w_s\rangle$ and $U = \langle u_k\rangle$. To show that $\ker\alpha = \ker\alpha^2$, let $v \in \ker\alpha^2$. Then $v = \sum a_sw_s + \sum b_iv_i + \sum c_ku_k$ for some scalars $a_s, b_i, c_k \in \mathbb{F}$. We obtain

$$0 = v\alpha^2 = \sum a_sw_s\alpha^2 + \sum b_iv_i\alpha^2 + \sum c_ku_k\alpha^2 = \sum b_iv_i\alpha^2 + \sum c_ku_k\alpha^2$$

since $w_s\alpha^2 = 0$ for all $s \in S$. Hence $b_i = 0$ for all $i \in I$ and $c_k = 0$ for all $k \in K$. Thus $v = \sum a_sw_s \in \ker\alpha$ which implies that $\ker\alpha^2 \subseteq \ker\alpha$. In general, $\ker\alpha \subseteq \ker\alpha^2$. Therefore, $\ker\alpha = \ker\alpha^2$ and so α is right regular by Theorem 3.6.

(ii) $\Rightarrow$ (iii). Assume that α is right regular. Then $\alpha \mathscr{R} \alpha^2$ which implies that $\alpha \mathscr{J} \alpha^2$ since $\mathscr{R} \subseteq \mathscr{J}$ in any semigroup. Hence α is intra-regular.

(iii) $\Rightarrow$ (i). Assume that α is intra-regular. Then $\dim(V\alpha/U) = \dim(V\alpha^2/U)$ by Theorem 3.7. Let $V\alpha^2 = \langle v_i\rangle \oplus U$. Since $V\alpha^2 \subseteq V\alpha$, we can write $V\alpha = \langle v_i\rangle \oplus \langle v_j\rangle \oplus U$. Moreover, since $\dim(V\alpha/U) = \dim(V\alpha^2/U)$ is finite, we have $|I| + |J| = |I|$ which implies that $|J| = 0$. Thus $V\alpha = \langle v_i\rangle \oplus U = V\alpha^2$. Therefore, α is left regular. □

We next provide necessary and sufficient conditions for $L_{GL(U)}(V)$ to be a left regular, right regular, intra-regular, or completely regular semigroup.

Theorem 3.9. *$L_{GL(U)}(V)$ is left regular if and only if* $\operatorname{codim}(U) \leq 1$.

Proof. Assume that $\operatorname{codim}(U) > 1$. We can write $V = W \oplus U$ where $\dim W > 1$. Then there exist nonzero vectors $v, w \in W$ such that $\{v, w\}$ is linearly independent. Let $W = \langle v, w\rangle \oplus \langle w_i\rangle$ and $U = \langle u_j\rangle$. Define $\alpha \in L_{GL(U)}(V)$ by

$$\alpha = \begin{pmatrix} w_i & w & v & u_j \\ 0 & 0 & w & u_j \end{pmatrix}.$$

It follows from Theorem 3.5 that α is not left regular since $V\alpha = \langle w\rangle \oplus U$ but $V\alpha^2 = U$.

Conversely, assume that $\operatorname{codim}(U) \leq 1$. If $\operatorname{codim}(U) = 0$, then $L_{GL(U)}(V) = GL(U)$ which is trivially left regular since it is a group. Now, suppose that $\operatorname{codim}(U) = 1$. Let $\alpha \in L_{GL(U)}(V)$.

If $V\alpha = U$, then $V\alpha^2 = U\alpha = U = V\alpha$ which implies that α is left regular. If $V\alpha = \langle v\alpha\rangle \oplus U$ for some nonzero vector $v\alpha \in V$, then $\ker\alpha = \{0\}$ since V is a direct sum of U and a one-dimensional subspace. Hence $\alpha \in GL(V)$ and so $V\alpha = V\alpha^2$. Therefore, $L_{GL(U)}(V)$ is left regular. □

Theorem 3.10. *$L_{GL(U)}(V)$ is right regular if and only if* $\operatorname{codim}(U) \leq 1$.

Proof. Assume that $\operatorname{codim}(U) > 1$. Define $\alpha \in L_{GL(U)}(V)$ as in the proof of Theorem 3.9. Then $v \in \ker\alpha^2$ but $v \notin \ker\alpha$. This implies that α is not right regular.

Conversely, assume that $\operatorname{codim}(U) \leq 1$. Let $\alpha \in L_{GL(U)}(V)$. Then $\dim(V\alpha/U) \leq \dim(V/U) = \operatorname{codim}(U) \leq 1$. By Theorem 3.8, α is right regular. Hence, $L_{GL(U)}(V)$ is right regular. □

Consider α defined as in the proof of Theorem 3.9, we have $\dim(V\alpha/U) = 1 \neq 0 = \dim(V\alpha^2/U)$. Hence α is also not intra-regular. From this fact with the same proof as given in Theorem 3.10, we obtain the following result.

Theorem 3.11. *$L_{GL(U)}(V)$ is intra-regular if and only if* $\operatorname{codim}(U) \leq 1$.

By combining Theorems 3.9 and 3.10, we obtain the following result immediately.

Theorem 3.12. *$L_{GL(U)}(V)$ is completely regular if and only if* $\operatorname{codim}(U) \leq 1$.

Recall that an element u in a monoid S with identity 1 is called a *unit* if there exists $u' \in S$ such that $uu' = 1 = u'u$. An element $a \in S$ is said to be *unit regular* if $a = aua$ for some unit $u \in S$. Furthermore, if every element of S is unit regular, then S is called a *unit regular semigroup*. Clearly, an element $\alpha \in L_{GL(U)}(V)$ is a unit if and only if it is bijective. In what follows, we provide a characterization of unit regular elements in $L_{GL(U)}(V)$.

Theorem 3.13. *Let $\alpha \in L_{GL(U)}(V)$. Then α is unit regular if and only if* $\operatorname{null}(\alpha) = \operatorname{codim}(V\alpha)$.

Proof. Assume that $\operatorname{null}(\alpha) = \operatorname{codim}(V\alpha)$. Let $V\alpha = \langle v_j\alpha\rangle \oplus U$. We can write

$$\alpha = \begin{pmatrix} w_i & v_j & u_k \\ 0 & v_j\alpha & u_k\alpha \end{pmatrix}$$

where $\ker\alpha = \langle w_i\rangle$ and $U = \langle u_k\rangle$. Since $\operatorname{null}(\alpha) = \operatorname{codim}(V\alpha)$, we can write $V = \langle v_i\rangle \oplus \langle v_j\alpha\rangle \oplus U$. Define a linear function $\beta : V \to V$ by

$$\beta = \begin{pmatrix} v_i & v_j\alpha & u_k\alpha \\ w_i & v_j & u_k \end{pmatrix}.$$

It is straightforward to verify that β is bijective such that $\alpha = \alpha\beta\alpha$. Hence α is unit regular.

Conversely, assume that α is unit regular. Then there exists a unit $\beta \in L_{GL(U)}(V)$ such that $\alpha = \alpha\beta\alpha$. Let $V\alpha = V\alpha\beta\alpha = \langle v_j\alpha\beta\alpha\rangle \oplus \langle u_k\alpha\beta\alpha\rangle$ where $U = \langle u_k\rangle$. Then $V = \langle w_i\rangle \oplus \langle v_j\alpha\beta\rangle \oplus \langle u_k\alpha\beta\rangle$ where $\ker\alpha = \langle w_i\rangle$. On the other hand, we have $V\alpha = \langle v_j\alpha\beta\alpha\rangle \oplus \langle u_k\alpha\beta\alpha\rangle = \langle v_j\alpha\rangle \oplus \langle u_k\alpha\rangle$ since $\alpha = \alpha\beta\alpha$. Let $V = \langle v_s\rangle \oplus \langle v_j\alpha\rangle \oplus \langle u_k\alpha\rangle$. Since β is a bijection, we can write $V = V\beta = \langle v_s\beta\rangle \oplus \langle v_j\alpha\beta\rangle \oplus \langle u_k\alpha\beta\rangle$. Then

$$\langle w_i\rangle \oplus \langle v_j\alpha\beta\rangle \oplus \langle u_k\alpha\beta\rangle = \langle v_s\beta\rangle \oplus \langle v_j\alpha\beta\rangle \oplus \langle u_k\alpha\beta\rangle$$

from which it follows that $\langle w_i\rangle \cong \langle v_s\beta\rangle$. Hence

$$\operatorname{null}(\alpha) = |I| = |S| = \dim(V/V\alpha) = \operatorname{codim}(V\alpha).$$

□

Remark 3.1. *For any unit regular element $\alpha \in L_{GL(U)}(V)$, we may express $V\alpha = \langle v_j\alpha\rangle \oplus U$. One can verify that such an element α satisfies conditions (i) and (ii) of* [11, Theorem 4.6], *and by choosing $T_\alpha = \langle v_j\rangle$, condition (iii) is also fulfilled. Consequently, the characterization of* *unit regular elements in $L_{GL(U)}(V)$ presented in Theorem 3.13 follows from* [11, Theorem 4.6]. *Nevertheless, we provide an alternative proof that is more direct and elementary.*

We next characterize when $L_{GL(U)}(V)$ is a unit regular semigroup. The proof follows immediately from [11, Theorem 4.8] and is therefore omitted.

Theorem 3.14. *$L_{GL(U)}(V)$ is a unit regular semigroup if and only if* $\operatorname{codim}(U)$ *is finite.*

For the remainder of this section, we assume that V is a finite n dimensional vector space over a finite field with q elements and U is a subspace of dimension r. Under these assumptions, the complement V/U is necessarily finite. By Theorem 3.14, every element of $L_{GL(U)}(V)$ is unit regular, which implies that the total number of unit regular elements equals $|L_{GL(U)}(V)|$. We can see that the order of $L_{GL(U)}(V)$ is the product of the order of $GL(U)$ and the number of linear transformations from W to V where W is a complement of U in V. Since $\dim W = n-r$, the number of linear transformations from W to V is $q^{n(n-r)}$. Therefore,

$$|L_{GL(U)}(V)| = |GL(U)|q^{n(n-r)}.$$

The order of $GL(U)$ will be discussed later in this section.

Nevertheless, $L_{GL(U)}(V)$ need not be left regular, right regular, or intra-regular as a semigroup. We conclude this section by enumerating the left regular, right regular, and intra-regular elements within $L_{GL(U)}(V)$. Since $\dim(V\alpha/U)$ is finite for all $\alpha \in L_{GL(U)}(V)$, Theorem 3.8 allows us to restrict our attention to counting only the left regular elements in this case.

Lemma 3.1. *Let $\dim V$ be finite. If $\alpha \in L_{GL(U)}(V)$ is left regular, then $V\alpha \cap \ker\alpha = \{0\}$.*

Proof. Assume that $\alpha \in L_{GL(U)}(V)$ is a left regular element. We obtain by Theorem 3.8 that α is right regular and hence $\ker\alpha = \ker\alpha^2$. Let $v\alpha \in V\alpha \cap \ker\alpha$. Then $v\alpha^2 = (v\alpha)\alpha = 0$ which implies that $v \in \ker\alpha^2 = \ker\alpha$. Thus $v\alpha = 0$ and so $V\alpha \cap \ker\alpha = \{0\}$. □

Theorem 3.15. *Let V be a finite dimensional vector space over a finite field. Then $\alpha \in L_{GL(U)}(V)$ is left regular if and only if $\alpha|_{V\alpha}$ is an automorphism.*

Proof. Assume that α is left regular. By Theorem 3.5, we have $V\alpha = V\alpha^2 = (V\alpha)\alpha$. Since V is a finite set, it follows that $\alpha|_{V\alpha}$ is bijective and hence an automorphism.

Conversely, assume that $\alpha|_{V\alpha}$ is an automorphism. Then $V\alpha = (V\alpha)\alpha = V\alpha^2$. By Theorem 3.5, α is left regular. □

For arbitrary positive integers n, k and q, we denote by $\binom{n}{k}_q$ the count of all k-dimensional subspaces contained in an n-dimensional vector space V over a finite field of q elements. This quantity $\binom{n}{k}_q$ is referred to as a *Gaussian binomial coefficient* (or *q-binomial coefficient*) and is given by

$$\binom{n}{k}_q = \frac{(q^n-1)\cdots(q-1)}{(q^k-1)\cdots(q-1)(q^{n-k}-1)\cdots(q-1)}$$

(further details can be found in [8], pages 57–58).

We employ the notation $GL(V)$ to represent the group of all automorphisms (invertible linear transformations) of a vector space V over a field $\mathbb{F}$, and $GL(n,q)$ to denote the group of all $n\times n$ invertible matrices over a field $\mathbb{F}$ with q elements. It is standard that $GL(V)$ forms a group under composition of functions, while $GL(n,q)$ forms a group under matrix multiplication with the identity matrix as the identity element. The group $GL(n,q)$ is known as the *general linear group of degree n*, and its order is computed as:

$$(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})$$

(see [9], pages 219–220 for further discussion).

When V is an n-dimensional vector space over a finite field $\mathbb{F}$ with $|\mathbb{F}| = q$, there exists an isomorphism between $GL(V)$ and $GL(n,q)$. Specifically, given a basis $\{v_1, v_2, \ldots, v_n\}$ of V and an automorphism $\alpha \in GL(V)$, we can express

$$v_k\alpha = \sum_{j=1}^{n} a_{jk}v_j$$

for suitable scalars $a_{jk} \in \mathbb{F}$. The matrix representation of α with respect to this basis is then the matrix whose entries are precisely a_{jk}.

By utilizing the concepts of Gaussian binomial coefficients and general linear groups, we can now determine the number of left regular elements in $L_{GL(U)}(V)$ when V is a finite-dimensional vector space over a finite field.

Theorem 3.16. *Let U be an r-dimensional subspace of a finite n-dimensional vector space V over a finite field $\mathbb{F}$ with q elements. Then the number of left regular elements in $L_{GL(U)}(V)$ is*

$$|GL(r,q)| \sum_{k=0}^{n-r} \binom{n-r}{k}_q q^{(k+r)(n-k-r)} |GL(k,q)|.$$

Proof. For each subspaces S and T of V such that $U \subseteq T$ and $V = S \oplus T$, define a subset $\mathfrak{L}(S,T)$ of $L_{GL(U)}(V)$ by

$$\mathfrak{L}(S,T) = \{\alpha \in L_{GL(U)}(V) : V\alpha = T \text{ and } \ker\alpha = S\}.$$

Let $\mathscr{P}$ be the set of all pairs (S,T) of subspaces of V such that $U \subseteq T$ and $V = S \oplus T$. It is easy to see that $\mathfrak{L}(S,T) \cap \mathfrak{L}(S',T') = \emptyset$ for all distinct pairs $(S,T), (S',T') \in \mathscr{P}$.

Now, we claim that $\bigcup_{(S,T)\in\mathscr{P}} \mathfrak{L}(S,T)$ is the set of all left regular elements in $L_{GL(U)}(V)$. Indeed, let $\alpha \in \mathfrak{L}(S,T)$. Then $V\alpha^2 = (V\alpha)\alpha = T\alpha$. Let $v \in V$. Then $v = s + t$ for some $s \in S$ and $t \in T$. We have $v\alpha = t\alpha \in T\alpha$ since $s\alpha = 0$. Hence $V\alpha \subseteq T\alpha \subseteq V\alpha$ which implies that $V\alpha = T\alpha = V\alpha^2$. Thus α is left regular by Theorem 3.5.

For each left regular element $\alpha \in L_{GL(U)}(V)$, since V is a finite dimensional vector space over a finite field, we obtain by Theorem 3.15 that $\alpha|_{V\alpha}$ is an automorphism. It is straightforward to verify that we can write

$$\alpha = \begin{pmatrix} w_i & v_j & u_k \\ 0 & v'_j & u'_k \end{pmatrix}$$

where $\ker\alpha = \langle w_i \rangle$, $\langle v_j \rangle = \langle v'_j \rangle$, and $U = \langle u_k \rangle = \langle u'_k \rangle$. Choose $S = \langle w_i \rangle$ and $T = \langle v_j \rangle \oplus \langle u_k \rangle = \langle v'_j \rangle \oplus \langle u'_k \rangle$. Then $V = S \oplus T$ and $\alpha \in \mathfrak{L}(S,T)$. Therefore, the claim holds. We conclude that the number of left regular elements in $L_{GL(U)}(V)$ is equal to $\sum_{(S,T)\in\mathscr{P}} |\mathfrak{L}(S,T)|$.

Let $(S,T) \in \mathscr{P}$. For each $\alpha \in \mathfrak{L}(S,T)$, we can write α as above. Since $\alpha|_{V\alpha}$ is an automorphism, the number of choices for $\alpha|_{V\alpha}$ is equal to $|GL(|J|,q)||GL(r,q)|$ where $|J| = \dim(T/U)$. Hence, we have $|\mathfrak{L}(S,T)| = |GL(\dim(T/U),q)||GL(r,q)|$.

To determine the number of pairs $(S,T) \in \mathscr{P}$, we first choose a subspace T of V such that $U \subseteq T$ and $\dim(T/U) = k$. The number of choices for such subspace T is equal to $\binom{n-r}{k}_q$. Next, for each chosen subspace T, we choose a subspace S of V such that $V = S \oplus T$. The number of choices for such subspace S is equal to the number of complements of T in V which is $q^{(k+r)(n-k-r)}$. Thus the number of pairs $(S,T) \in \mathscr{P}$ such that $\dim(T/U) = k$ is equal to $\binom{n-r}{k}_q q^{(k+r)(n-k-r)}$. Therefore, the number of pairs $(S,T) \in \mathscr{P}$ is

$$\sum_{k=0}^{n-r} \binom{n-r}{k}_q q^{(k+r)(n-k-r)}.$$

This completes the proof. □

The results in this section are very close to those for $PG_Y(X)$ in [1]. In the linear case, $\dim(V\alpha/U)$ plays the same role as $|X\alpha \setminus Y|$ in the set case. Also, the condition $\operatorname{codim}(U) \leq 1$ for left/right/intra/completely regularity matches the condition $|X \setminus Y| \leq 1$ for $PG_Y(X)$. Likewise, $\operatorname{codim}(U) < \infty$ for unit regularity matches $|X \setminus Y| < \infty$. Finally, our counting formula extends the counting method in [1] from finite sets to finite-dimensional vector spaces over finite fields.

4. Maximal subsemigroups

In this section, we investigate the maximal subsemigroups of the semigroup $L_{GL(U)}(V)$. We begin by recalling some useful results from [10].

In [10], the author gave a characterization of the proper ideals of $L_{GL(U)}(V)$ as follows.

Theorem 4.17. *The proper ideals of $L_{GL(U)}(V)$ are precisely the sets*

$$Q(k) = \{\alpha \in L_{GL(U)}(V) : \dim(V\alpha/U) < k\}$$

where $1 \leq k \leq \operatorname{codim}(U)$.

For each $0 \leq k \leq \operatorname{codim}(U)$, define a subset $J(k)$ of $L_{GL(U)}(V)$ by

$$J(k) = \{\alpha \in L_{GL(U)}(V) : \dim(V\alpha/U) = k\}.$$

It follows by [10] that $J(k)$ is a $\mathscr{J}$-class of $L_{GL(U)}(V)$. We can see that the number of $\mathscr{J}$-classes of $L_{GL(U)}(V)$ is $\operatorname{codim}(U)+1$. Obviously, if $\dim V$ is finite, then

$$Q(k) = J(0)\dot{\cup}J(1)\dot{\cup}J(2)\dot{\cup}\cdots\dot{\cup}J(k-1).$$

Let $GL(V,U) = \{\alpha \in GL(V) : \alpha|_U \in GL(U)\}$. It is easy to verify that $GL(V,U)$ is a subgroup of $GL(V)$. Moreover, the following proposition shows that this subgroup is the group of units of $L_{GL(U)}(V)$. The proof of this fact is straightforward, so we will omit it.

Proposition 4.1. *$GL(V,U)$ is the group of units of $L_{GL(U)}(V)$.*

Throughout this section, let V be a finite n-dimensional vector space over a finite field $\mathbb{F}$ and U an r-dimensional subspace of V. From now on, we write

$$U = \langle u_1, u_2, \ldots, u_r \rangle.$$

For any subset T of a semigroup S, we denote by $\langle T \rangle$ the subsemigroup of S generated by T, i.e., the smallest subsemigroup of S containing T.

Now, we present the following useful results from [10].

Lemma 4.2. [10, Lemma 5.4] *Let $\alpha \in J(n-r-1)$. Then $L_{GL(U)}(V) = \langle J(n-r) \cup \{\alpha\} \rangle$.*

Theorem 4.18. [10, Theorem 5.7] *$J(n-r)$ is a subgroup of $L_{GL(U)}(V)$.*

We are now in a position to describe some maximal subsemigroups of $L_{GL(U)}(V)$.

Theorem 4.19. *Let M be a maximal subgroup of $J(n-r)$. Then the following hold:*

(i) *$M \cup Q(n-r)$ is a maximal subsemigroup of $L_{GL(U)}(V)$.*

(ii) *$M \cup Q(n-r)$ is a maximal regular subsemigroup of $L_{GL(U)}(V)$.*

Proof. (i) Since M is a subgroup and $Q(n-r)$ is an ideal, we have $M \cup Q(n-r)$ is a subsemigroup of $L_{GL(U)}(V)$. To show that $M \cup Q(n-r)$ is maximal, suppose that $M \cup Q(n-r) \subsetneq S$ for some subsemigroup S of $L_{GL(U)}(V)$. Then there exists $\gamma \in S \setminus (M \cup Q(n-r))$. We obtain the subgroup of $J(n-r)$ generated by $M \cup \{\gamma\}$ is $J(n-r)$ since M is maximal. Hence $J(n-r) \subseteq S$. Moreover, since $J(n-r-1) \subseteq Q(n-r)$, it follows from Lemma 4.2 and Theorem 4.18 that $L_{GL(U)}(V) = \langle J(n-r) \cup J(n-r-1) \rangle \subseteq S$. Therefore, $S = L_{GL(U)}(V)$, which implies that $M \cup Q(n-r)$ is maximal.

(ii) It suffices to show $M \cup Q(n-r)$ is regular. Let $\alpha \in M \cup Q(n-r)$. If $\alpha \in M$, then α is regular since M is a group. If $\alpha \in Q(n-r)$, then $\alpha = \alpha\beta\alpha$ for some $\beta \in L_{GL(U)}(V)$. We can write $\alpha = \alpha(\beta\alpha\beta)\alpha$ where $\beta\alpha\beta \in Q(n-r)$ since $Q(n-r)$ is an ideal. Thus, α is regular. Therefore, $M \cup Q(n-r)$ is a maximal regular subsemigroup of $L_{GL(U)}(V)$. □

Theorem 4.20. *The following statements hold.*

(i) $J(n-r) \cup Q(n-r-1)$ *is a maximal subsemigroup of* $L_{GL(U)}(V)$.
(ii) $J(n-r) \cup Q(n-r-1)$ *is a maximal regular subsemigroup of* $L_{GL(U)}(V)$.

Proof. (i) Since $J(n-r)$ is a subgroup and $Q(n-r-1)$ is an ideal, we have $J(n-r) \cup Q(n-r-1)$ is a subsemigroup of $L_{GL(U)}(V)$. To show that $J(n-r) \cup Q(n-r-1)$ is maximal, let $\alpha \in L_{GL(U)}(V) \setminus (J(n-r) \cup Q(n-r-1))$. Then $\alpha \in J(n-r-1)$. By Lemma 4.2, we have $L_{GL(U)}(V) = \langle J(n-r) \cup \{\alpha\}\rangle$. Hence $J(n-r) \cup Q(n-r-1)$ is a maximal subsemigroup of $L_{GL(U)}(V)$. The proof of (ii) is similar to that of (ii) in Theorem 4.19. □

Lemma 4.3. *Let S be a subset of $L_{GL(U)}(V)$. Then the following statements hold.*

(i) *If S is a maximal subsemigroup of $L_{GL(U)}(V)$, then either $J(n-r) \subseteq S$ or $J(n-r-1) \subseteq S$.*
(ii) *If S is a maximal regular subsemigroup of $L_{GL(U)}(V)$, then either $J(n-r) \subseteq S$ or $J(n-r-1) \subseteq S$.*

Proof. (i) Assume that S is a maximal subsemigroup of $L_{GL(U)}(V)$. Then $S \cap J(n-r)$ is a subgroup of $J(n-r)$. If $J(n-r) \nsubseteq S$, then $S \cap J(n-r) \subsetneq J(n-r)$. There exists a maximal subgroup M of $J(n-r)$ such that $S \cap J(n-r) \subseteq M$. We can see that $S \subseteq M \cup Q(n-r)$. By (i) of Theorem 4.19, $M \cup Q(n-r)$ is a maximal subsemigroup of $L_{GL(U)}(V)$. Hence, $S = M \cup Q(n-r)$ which implies that $J(n-r-1) \subseteq S$.

The proof of (ii) is similar to that of (i). □

To prove our main result in this section, we need the following corollary.

Corollary 4.1. [10, Corollary 5.2] $Q(k+1) = \langle J(k)\rangle$ *for all* $1 \le k \le n-r-1$.

The next lemma is a generalization of [10, Lemma 5.3] and is also required for our main result.

Lemma 4.4. *Let k be a natural number such that $1 \le k \le n-r$. Then $J(n-r-k) \subseteq J(n-r)\alpha J(n-r)$ for all $\alpha \in J(n-r-k)$*

Proof. Let $\alpha \in J(n-r-k)$. Let

$$V\alpha = \langle v_1\alpha, v_2\alpha, \ldots, v_{n-r-k}\alpha\rangle \oplus U \text{ and } \ker\alpha = \langle v_{n-r-k+1}, v_{n-r-k+2}, \ldots, v_{n-r}\rangle.$$

We can write

$$\alpha = \begin{pmatrix} v_1 & v_2 & \cdots & v_{n-r-k} & v_{n-r-k+1} & v_{n-r-k+2} & \cdots & v_{n-r} & u_i \\ v_1\alpha & v_2\alpha & \cdots & v_{n-r-k}\alpha & 0 & 0 & \cdots & 0 & u_i\alpha \end{pmatrix}$$

where $U = \langle u_i\rangle$ and $i \in \{1,2,\ldots,r\}$. Let $\beta \in J(n-r-k)$. Similar to α, we can write

$$\beta = \begin{pmatrix} w_1 & w_2 & \cdots & w_{n-r-k} & w_{n-r-k+1} & w_{n-r-k+2} & \cdots & w_{n-r} & u_i \\ w_1\beta & w_2\beta & \cdots & w_{n-r-k}\beta & 0 & 0 & \cdots & 0 & u_i\beta \end{pmatrix}.$$

Let

$$V = \langle v_1\alpha, v_2\alpha, \ldots, v_{n-r-k}\alpha\rangle \oplus \langle s_1, s_2, \ldots, s_k\rangle \oplus U$$

and

$$V = \langle w_1\beta, w_2\beta, \ldots, w_{n-r-1}\beta\rangle \oplus \langle s_1', s_2', \ldots, s_k'\rangle \oplus U.$$

Define $\lambda, \mu \in L_{GL(U)}(V)$ by

$$\lambda = \begin{pmatrix} w_1 & w_2 & \cdots & w_{n-r-k} & w_{n-r-k+1} & w_{n-r-k+2} & \cdots & w_{n-r} & u_i \\ v_1 & v_2 & \cdots & v_{n-r-k} & v_{n-r-k+1} & v_{n-r-k+2} & \cdots & v_{n-r} & u_i \end{pmatrix}$$

and

$$\mu = \begin{pmatrix} v_1\alpha & v_2\alpha & \cdots & v_{n-r-k}\alpha & s_1 & s_2 & \cdots & s_k & u_i\alpha \\ w_1\beta & w_2\beta & \cdots & w_{n-r-k}\beta & s_1' & s_2' & \cdots & s_k' & u_i\beta \end{pmatrix}.$$

It is straightforward to verify that $\beta = \lambda\alpha\mu$ and $\lambda, \mu \in J(n-r)$. □

Remark 4.2. *Lemma 4.4 extends* [10, Lemma 5.3]. *In* [10, Lemma 5.3], *one step down is proved (case $k = 1$). Our lemma covers all steps: $k = 1, 2, \ldots, n-r$. So the earlier lemma is a special case ($k = 1$).*

Now we are ready to present our main results in this section.

Theorem 4.21. *Let S be a maximal subsemigroup of $L_{GL(U)}(V)$. Then one of the following holds.*

(i) *$S = M \cup Q(n-r)$ for some maximal subgroup M of $J(n-r)$.*
(ii) *$S = J(n-r) \cup Q(n-r-1)$.*

Proof. We consider the following two cases.

Case 1: $S \cap J(n-r-1) \neq \emptyset$. Then there is $\alpha \in S \cap J(n-r-1)$. If $J(n-r) \subseteq S$, then by Lemma 4.2, we have $L_{GL(U)}(V) = \langle J(n-r) \cup \{\alpha\} \rangle \subseteq S$ which implies that $S = L_{GL(U)}(V)$. This contradicts the fact that S is a maximal subsemigroup of $L_{GL(U)}(V)$. Then $J(n-r) \nsubseteq S$ which implies that $S \cap J(n-r) \subsetneq J(n-r)$. We have $S \cap J(n-r)$ is a proper subgroup of $J(n-r)$. There exists a maximal subgroup M of $J(n-r)$ such that $S \cap J(n-r) \subseteq M$. We can see that $S \subseteq M \cup Q(n-r)$. By (i) of Theorem 4.19, $M \cup Q(n-r)$ is a maximal subsemigroup of $L_{GL(U)}(V)$. Hence, $S = M \cup Q(n-r)$.

Case 2: $S \cap J(n-r-1) = \emptyset$. By Lemma 4.3 (i), we have $J(n-r) \subseteq S$. If $S \cap Q(n-r-1) = \emptyset$, then it must be that $S = J(n-r)$. However, the situation in which $S \cap Q(n-r-1) = \emptyset$ cannot occur, because $J(n-r) \subseteq J(n-r) \cup Q(n-r-1)$ and $J(n-r) \cup Q(n-r-1)$ is a maximal subsemigroup of $L_{GL(U)}(V)$. Hence $S \cap Q(n-r-1)$ is nonempty. Let k be the smallest natural number such that $S \cap Q(n-r-1) \cap J(n-r-k) \neq \emptyset$. We note that $k \geq 2$. Let $\alpha \in S \cap Q(n-r-1) \cap J(n-r-k)$. By Lemma 4.4, we have $J(n-r-k) \subseteq J(n-r)\alpha J(n-r) \subseteq S$. Moreover, by Corollary 4.1, we obtain $Q(n-r-k+1) = \langle J(n-r-k) \rangle \subseteq S$. It is concluded that $S = Q(n-r-k+1) \cup J(n-r) \subseteq Q(n-r-1) \cup J(n-r)$. Since S and $Q(n-r-1) \cup J(n-r)$ are maximal subsemigroups of $L_{GL(U)}(V)$, we have $S = J(n-r) \cup Q(n-r-1)$. □

By the same argument as in the proof of Theorem 4.21, we can show that the following theorem holds.

Theorem 4.22. *Let S be a maximal regular subsemigroup of $L_{GL(U)}(V)$. Then one of the following holds.*

(i) *$S = M \cup Q(n-r)$ for some maximal subgroup M of $J(n-r)$.*
(ii) *$S = J(n-r) \cup Q(n-r-1)$.*

We now compare our maximal subsemigroup results with [1]. For $PG_Y(X)$, there are three types of maximal subsemigroups. For $L_{GL(U)}(V)$, there are only two types. The first type is $M \cup Q(n-r)$, where M is a maximal subgroup of $J(n-r)$. This corresponds to the first type in $PG_Y(X)$. In our linear setting, the second and third types from $PG_Y(X)$ merge into one type, namely $J(n-r) \cup Q(n-r-1)$.

Even with this difference, both semigroups share one key property: their maximal subsemigroups are exactly their maximal regular subsemigroups.

CONCLUDING REMARKS

In this paper, we studied regularity and maximal subsemigroups of $L_{GL(U)}(V)$ in detail. This continues the work in [10].

In Section 3, we described left regular, right regular, intra-regular, and unit regular elements. We also gave necessary and sufficient conditions for $L_{GL(U)}(V)$ to have each property. When V is finite-dimensional over a finite field, we counted left regular elements explicitly.

In Section 4, we classified all maximal subsemigroups and all maximal regular subsemigroups of $L_{GL(U)}(V)$, and proved that these two classes are the same.

Our results are parallel to those for $PG_Y(X)$ [1]: $\operatorname{codim}(U)$ in the linear setting corresponds to $|X \setminus Y|$ in the set setting. But there is also a difference: $PG_Y(X)$ has three maximal-subsemigroup types, while $L_{GL(U)}(V)$ has two.

Our work also supports the general framework of [11] for $L_{\mathbb{S}(W)}(V)$. By taking $\mathbb{S}(W) = GL(U)$, we obtain explicit formulas and a full maximal-subsemigroup classification.

A natural next step is to consider other choices of $\mathbb{S}(W)$, for example semigroups of injective linear maps.

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