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# Asymptotic solutions of singularly perturbed differential equations with piecewise constant argument

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ABSTRACT. This paper examines the Cauchy problem for a singularly perturbed differential equation that has a piecewise constant argument. Various types of singular perturbation problems have been discussed in the literature. Notably, this study considers an extension of Tikhonov's theorem for singularly perturbed differential equations with a piecewise-constant generalized argument.

In our current research, we construct a uniform asymptotic approximation for solutions that is valid over the entire closed interval, utilizing the boundary function method. We develop an asymptotic expansion for the solution of a singularly perturbed initial problem, achieving an arbitrary degree of accuracy for a small parameter. Additionally, we present an algorithm that helps determine the asymptotic terms associated with the solution.

We formulate a theorem to estimate the residual term of the asymptotic expansion, which quantifies the difference between the exact solution and its approximation. Furthermore, we observe a humping phenomenon within the boundary layer. Lastly, an illustrative example showcasing modeling is provided.

## 1. INTRODUCTION

The interest in singular perturbation equations arises from their capability to model a wide range of applied problems in fields such as diffusion, chemical kinetics, biology, physics, hydrodynamics, and engineering. Various types of singular perturbation problems have been explored in the literature [1, 2, 3]. These problems are characterized by a small parameter that leads to non-uniform behavior of the solutions as this parameter approaches zero. Several asymptotic methods exist for approximating the solutions of singularly perturbed problems, allowing for uniform approximations with a specified accuracy.

Research conducted in [1, 2] has introduced the boundary function method, also known as the boundary layer correction method [3]. This method can effectively solve singularly perturbed problems, provided the conditions of the well-known Tikhonov theorem are met in a particular domain. A range of asymptotic methods is discussed in [3].

Additionally, in [4, 5, 6, 7], impulsive systems that exhibit singularities are examined. The main innovation here is that, besides the singular nature of the differential equation, the impulsive equation itself is also singular. The boundary function method is employed to derive the principal result. This paper further utilizes the boundary layer method to analyze singular differential equations with piecewise constant coefficients.

The study of differential equations with piecewise constant argument (EPCA) of the form

$$\frac{dx(t)}{dt} = f(t, x(t), x([t])), \tag{1.1}$$

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where $[\cdot]$ denotes the greatest integer function, was originally introduced in [8] and has subsequently attracted significant attention in the literature [9, 10, 11]. Such equations are regarded as hybrid systems, since they exhibit features of both continuous dynamics and discrete-time processes. Notably, even a one-dimensional logistic equation of this kind can demonstrate chaotic behavior[12].

Differential equations with piecewise-constant arguments (EPCA) are widely used in biomedicine, chemistry, mechanical engineering, and physics due to their ability to model processes that combine continuous and discrete dynamics. The first mathematical formulation of this kind was introduced by Busenberg and Cooke [13] for epidemiological models, incorporating mortality and transmission rates. Subsequent research has focused on the qualitative behavior of EPCA, including existence, oscillation, and stability of solutions [14, 15, 16].

Differential equations with generalized piecewise constant argument (EPCAG) were introduced in 2005 by M. Ahmet [17, 18] and subsequently studied in [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. These systems have attracted the attention of scientists for their efficiency in physics, economics, and biology. Differential equations with piecewise constants (EPCA) are very useful as models for neural networks. This was shown in [27, 28, 29], where the authors used EPCA. More detailed information on these systems can be found in the book [19]. They contain, as a subclass, piecewise constant argument differential equations. In [17], the argument functions were generalized, and it was proposed to reduce the study of EPCAG to integral equations. The proposals were not only very general in a modeling sense, but also very powerful in a methodological sense, since equivalent integral equations opened the door to investigations using methods of operator theory and functional analysis. Impressive studies of ordinary differential equations, impulsive differential equations, functional differential equations, and partial differential equations followed these suggestions. In mathematics, the generalization of piecewise constants unifies equations with retarded (delayed) and advanced arguments, thereby extending their applicability. Differential equations with functional response on the piecewise constant argument were first introduced in the paper [30].

In paper [31], a singularly perturbed initial problem for a second-order linear differential equation with piecewise constant argument is considered. An initial problem for first-order linear differential equations with piecewise-constant argument is formulated, defining the regular and boundary-layer terms. An asymptotic estimate for the residual term of the solution of the Cauchy problem is obtained. Uniform asymptotic solutions are constructed using the boundary layer method.

Tikhonov-type theorems express the limiting behavior of solutions of the singularly perturbed system [32, 33, 34]. It is a powerful tool for analyzing singular perturbation problems. In paper [35], an extension of Tikhonov's theorem for a singularly perturbed differential equation with a piecewise constant argument of generalized type has been considered. An approximate solution of the problem has been obtained. But this solution is only a zero approximation. In the boundary layer region, a new phenomenon, a hump, has been observed. In the present paper, we construct a higher-accuracy approximation and the complete asymptotic expansion for solutions of singularly perturbed differential equations with piecewise-constant argument.

## 2. Formalities of approximation

In this section of the paper, we demonstrate the algorithm for constructing an asymptotic approximation of the solution to singularly perturbed differential equations with a piecewise constant argument using the method of boundary functions. Let us consider the following system

$$\begin{cases} \varepsilon\dot{x} = F(x, y(\beta(t))), \\ \varepsilon\dot{y} = Q(x, y), \end{cases} \tag{2.2}$$

with initial condition

$$x(0,\varepsilon) = x_0, \quad y(0,\varepsilon) = \psi, \tag{2.3}$$

where $\varepsilon$ is a small positive number. The piecewise constant argument is determined by the function $\beta(t) = \theta_i$, if $t \in [\theta_i, \theta_{i+1}), i = 1, 2, ..., p,\ 0 = \theta_1 < \theta_2 < ... < \theta_p < T$, where $T$ is a fixed positive number, $\theta_i, i = 1, 2, ..., p$ are fixed numbers.

The following assumptions are required throughout the paper:

$(C1)$ The functions $F$ and $Q$ are continuously differentiable in the interior of the domain $G = \{(x, y, t),\ \|x\| \leq a,\ \|y\| \leq a, 0 \leq t \leq T\}$, where $a$ is a fixed positive number.

There exists a point $(\varphi, \psi)$ in the domain such that

$(C2)$ The function $F$ satisfies the conditions:

$$F(\varphi, \psi) = 0,\ \ F_x(\varphi, \psi) < 0,\ \ F_y(\varphi, \psi) < 0.$$

$(C3)$ The function $Q$ satisfies the conditions:

$$Q(\varphi, \psi) = 0,\ \ Q_x(\varphi, \psi) < 0,\ \ Q_y(\varphi, \psi) < 0.$$

We will seek for a asymptotic representation of the solution $x(t,\varepsilon), y(t,\varepsilon)$ of problem (2.2), (2.3) in the form

$$\begin{aligned} x(t,\varepsilon) &= \tilde{x}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon),\ \ \tau_i = \frac{t-\theta_i}{\varepsilon},\ \ i = 1, 2, ..., p, \\ y(t,\varepsilon) &= \tilde{y}(t,\varepsilon) + \nu^{(i)}(\tau_i,\varepsilon),\ \ \theta_i \leq t < \theta_{i+1}, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \tilde{x}(t,\varepsilon) &= \textstyle\sum_{k=0}^{\infty} \varepsilon^k \tilde{x}_k(t), \qquad \tilde{y}(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \tilde{y}_k(t), \\ \omega^{(i)}(\tau_i,\varepsilon) &= \textstyle\sum_{k=0}^{\infty} \varepsilon^k \omega_k^{(i)}(\tau_i), \qquad \nu^{(i)}(\tau_i,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \nu_k^{(i)}(\tau_i). \end{aligned} \tag{2.5}$$

Substituting the series (2.4) into system (2.2), we obtain the equalities

$$\begin{aligned} &\varepsilon(\tilde{x}'(t,\varepsilon) + \frac{1}{\varepsilon}\dot{\omega}^{(i)}(\tau_i,\varepsilon)) = F(\tilde{x}(t,\varepsilon), \tilde{y}(\beta(t),\varepsilon)) \\ &+ [F(\tilde{x}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \tilde{y}(\beta(t),\varepsilon) + \nu^{(i)}(0,\varepsilon)) - F(\tilde{x}(t,\varepsilon), \tilde{y}(\beta(t),\varepsilon))], \\ &\varepsilon(\tilde{y}'(t,\varepsilon) + \frac{1}{\varepsilon}\dot{\nu}^{(i)}(\tau_i,\varepsilon)) = Q(\tilde{x}(t,\varepsilon), \tilde{y}(t,\varepsilon)) \\ &+ [Q(\tilde{x}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \tilde{y}(t,\varepsilon) + \nu^{(i)}(\tau_i,\varepsilon)) - Q(\tilde{x}(t,\varepsilon), \tilde{y}(t,\varepsilon))]. \end{aligned}$$

Equalizing expressions for $t$ and $\tau_i$ in the last equations, we get two systems

$$\varepsilon\tilde{x}'(t,\varepsilon) = F(\tilde{x}(t,\varepsilon), \tilde{y}(\beta(t),\varepsilon)),\ \varepsilon\tilde{y}'(t,\varepsilon) = Q(\tilde{x}(t,\varepsilon), \tilde{y}(t,\varepsilon)), \tag{2.6}$$

and

$$\begin{aligned} \dot{\omega}^{(i)}(\tau_i,\varepsilon) &= F(\tilde{x}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \tilde{y}(\beta(t),\varepsilon) + \nu^{(i)}(0,\varepsilon)) - F(\tilde{x}(t,\varepsilon), \tilde{y}(\beta(t),\varepsilon)), \\ \dot{\nu}^{(i)}(\tau_i,\varepsilon) &= Q(\tilde{x}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \tilde{y}(t,\varepsilon) + \nu^{(i)}(\tau_i,\varepsilon)) - Q(\tilde{x}(t,\varepsilon), \tilde{y}(t,\varepsilon)). \end{aligned} \tag{2.7}$$

Now we represent $F$, $Q$ in the form of power series in $\varepsilon$,

$$\begin{aligned} &F(\tilde{x}(t,\varepsilon), \tilde{y}(\beta(t),\varepsilon)) = F(\tilde{x}_0(t) + \varepsilon\tilde{x}_1(t) + \ldots, \tilde{y}_0(\theta_i) + \varepsilon\tilde{y}_1(\theta_i) + \ldots) \\ &= F(\tilde{x}_0(t), \tilde{y}_0(\theta_i)) + \varepsilon[F_x(t)\tilde{x}_1(t) + F_y(t)\tilde{y}_1(\theta_i)] + \ldots \\ &+ \varepsilon^k[F_x(t)\tilde{x}_k(t) + F_y(t)\tilde{y}_k(\theta_i) + F_k(t)] + \ldots = \tilde{F}_0(t) + \varepsilon\tilde{F}_1(t) + \ldots \varepsilon^k\tilde{F}_k(t) + \ldots, \end{aligned}$$

where functions $F_x(t)$, and $F_y(t)$ are calculated at $(\tilde z_0(t), \tilde y_0(\theta_i))$ and $F_k(t)$ are expressed recursively through $\tilde x_j(t)$ and $\tilde y_j(t)$ with $j < k$. Similar expansions hold for $Q(\tilde x(t,\varepsilon), \tilde y(t,\varepsilon))$ :

$$\begin{aligned}
&F(\tilde x(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \tilde y(\beta(t),\varepsilon) + \nu^{(i)}(0,\varepsilon)) - F(\tilde x(t,\varepsilon), \tilde y(\theta_i,\varepsilon))\\
&= F(\tilde x_0(\theta_i+\varepsilon\tau_i) + \varepsilon\tilde x_1(\theta_i+\varepsilon\tau_i) + \ldots + \omega_0^{(i)}(\tau_i) + \varepsilon\omega_1^{(i)}(\tau_i) + \ldots,\\
&\tilde y_0(\theta_i) + \varepsilon\tilde y_1(\theta_i) + \ldots + \nu_0^{(i)}(0) + \varepsilon\nu_1^{(i)}(0) + \ldots)\\
&- F(\tilde x_0(\theta_i+\varepsilon\tau_i) + \varepsilon\tilde x_1(\theta_i+\varepsilon\tau_i) + \ldots, \tilde y_0(\theta_i) + \varepsilon\tilde y_1(\theta_i) + \ldots)\\
&= F(\tilde x_0(\theta_i) + \omega_0^{(i)}(\tau_i), \tilde y_0(\theta_i) + \nu_0^{(i)}(0)) - F(\tilde x_0(\theta_i), \tilde y_0(\theta_i))\\
&+ \varepsilon[F_x(\tau_i)\omega_1^{(i)}(\tau_i) + F_y(\tau_i)\nu_1^{(i)}(0) + T_1(\tau_i)] + \ldots\\
&+ \varepsilon^k[F_x(\tau_i)\omega_k^{(i)}(\tau_i) + F_y(\tau_i)\nu_k^{(i)}(0) + T_k(\tau_i)] + \ldots\\
&+ \Pi_0F(\tau_i) + \varepsilon\Pi_1F(\tau_i) + \ldots + \varepsilon^k\Pi_kF(\tau_i) + \ldots,
\end{aligned}$$

where the elements $F_z(\tau_i)$ and $F_y(\tau_i)$ are calculated at $(\tilde z_0(\theta_i)+\omega_0^{(i)}(\tau_i), \tilde y_0(\theta_i)+\nu_0^{(i)}(0)),\ i = 1,2,...,p$, the elements $F_z(\theta_i)$ and $F_y(\theta_i)$ are calculated at $(\tilde z_0(\theta_i), \tilde y_0(\theta_i))$ and $T_k(\tau_i),\ i = 1,2,...,p$, are expressed recursively through $\omega_j^{(i)}(\tau_i)$ and $\nu_j^{(i)}(0)$ with $j < k$.

Now the problems (2.2), (2.3) and (2.6), (2.7) take the form

$$\begin{aligned}
\varepsilon(\tilde x_0'(t) + \varepsilon\tilde x_1'(t) + \ldots + \varepsilon^k\tilde x_k'(t) + \ldots) &= \tilde F_0(t) + \varepsilon\tilde F_1(t) + \ldots\varepsilon^k\tilde F_k(t) + \ldots,\\
\varepsilon(\tilde y_0'(t) + \varepsilon\tilde y_1'(t) + \ldots + \varepsilon^k\tilde y_k'(t) + \ldots) &= \tilde Q_0(t) + \varepsilon\tilde Q_1(t) + \ldots\varepsilon^k\tilde Q_k(t) + \ldots,\\
\dot\omega_0^{(i)}(\tau_i) + \varepsilon\dot\omega_1^{(i)}(\tau_i) + \ldots + \varepsilon^k\dot\omega_k^{(i)}(\tau_i) + \ldots &= \Pi_0F(\tau_i) + \varepsilon\Pi_1F(\tau_i) + \ldots + \varepsilon^k\Pi_kF(\tau_i)\ldots,\\
\dot\nu_0^{(i)}(\tau_i) + \varepsilon\dot\nu_1^{(i)}(\tau_i) + \ldots + \varepsilon^k\dot\nu_k^{(i)}(\tau_i) + \ldots &= \Pi_0Q(\tau_i) + \varepsilon\Pi_1Q(\tau_i) + \ldots + \varepsilon^k\Pi_kQ(\tau_i) + \ldots
\end{aligned}$$

Substituting the expansion (2.5) into conditions (2.3), we obtain

$$x(0,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k\tilde x_k(0) + \sum_{k=0}^{\infty}\varepsilon^k\omega_k^{(0)}(0) = x_0,$$

and

$$y(0,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k\tilde y_k(0) + \sum_{k=0}^{\infty}\varepsilon^k\nu_k^{(0)}(0) = \psi.$$

In fact, the above expansions up to order $n$. We equate the coefficients according to the powers of $\varepsilon$. To determine the approximation of order zero $\tilde x_0(t), \tilde y_0(t), \omega_0^{(i)}(\tau_i)$ and $\nu_0^{(i)}(\tau_i), i = 1,2,...,p$, we obtain the systems

$$\begin{cases} 0 = F(\tilde x_0(t), \tilde y_0(\theta_i)),\\ 0 = Q(\tilde x_0(t), \tilde y_0(t)),\end{cases}$$

$$\begin{cases} \dot\omega_0^{(i)}(\tau_i) = F(\tilde x_0(\theta_i) + \omega_0^{(i)}(\tau_i), \tilde y_0(\theta_i) + \nu_0^{(i)}(0)) - F(\tilde x_0(\theta_i), \tilde y_0(\theta_i)) = \Pi_0F(\tau_i),\\ \dot\nu_0^{(i)}(\tau_i) = Q(\tilde x_0(\theta_i) + \omega_0^{(i)}(\tau_i), \tilde y_0(\theta_i) + \nu_0^{(i)}(\tau_i)) - Q(\tilde x_0(\theta_i), \tilde y_0(\theta_i)) = \Pi_0Q(\tau_i),\end{cases} \tag{2.8}$$

To define the coefficients of $\varepsilon^k (k \geq 1)$, we use the equations

$$\begin{cases} \tilde x_{k-1}'(t) = F_x(t)\tilde x_k(t) + F_y(t)\tilde y_k(\theta_i) + F_k(t),\\ \tilde y_{k-1}'(t) = Q_x(t)\tilde x_k(t) + Q_y(t)\tilde y_k(t) + Q_k(t),\end{cases}$$

$$\begin{cases} \dot\omega_k^{(i)}(\tau_i) = F_x(\tau_i)\omega_k^{(i)}(\tau_i) + F_y(\tau_i)\nu_k^{(i)}(0) + G_k(\tau_i) = \Pi_kF(\tau_i),\\ \dot\nu_k^{(i)}(\tau_i) = Q_x(\tau_i)\omega_k^{(i)}(\tau_i) + Q_y(\tau_i)\nu_k^{(i)}(\tau_i) + T_k(\tau_i) = \Pi_kQ(\tau_i).\end{cases}$$

Consider the interval $t \in [0, \theta_1]$. To find the coefficients of $\varepsilon^0$ for the approximation $\tilde{x}_0(t)$ and $\tilde{y}_0(t)$ we have the system

$$\begin{cases} 0 = F(\tilde{x}_0(t), \tilde{y}_0(0)), \\ 0 = Q(\tilde{x}_0(t), \tilde{y}_0(t)). \end{cases}$$

In view of conditions $(C2)$ and $(C3)$ we choose $\tilde{x}_0(t) = \tilde{x}(t) = \varphi(t), \tilde{y}_0(t) = \tilde{y}(t) = \psi(t)$. The first equation of (2.8), according to the mean value theorem, can be written in the form

$$\dot{\omega}_0^{(0)}(\tau_0) = F_x(\tilde{x}_0(0) + \Theta\omega_0^{(0)}(\tau_0), \tilde{y}_0(0))\omega_0^{(0)}(\tau_0).$$

where $0 < \Theta < 1$, where $\Theta$ is a number. From the last equation and initial condition

$$\omega_0^{(0)}(0) = x_0 - \tilde{x}_0(0)$$

one can find

$$\omega_0^{(0)}(\tau_0) = (x_0 - \tilde{x}_0(0)) \exp\left( \int_0^{\tau_0} F_x(\tilde{x}_0(0) + \Theta\omega_0^{(0)}(s), \tilde{y}_0(0))ds \right). \tag{2.9}$$

In view of condition $(C2)$, $\omega_0^{(0)}(\tau_0)$ possesses the exponential estimate

$$|\omega_0^{(0)}(\tau_0)| \le c \exp(-\kappa\tau_0),$$

where $c$ and $\kappa$ are positive numbers.

Next, we represent the second equation of (2.8) in the form

$$\dot{\nu}_0^{(0)}(\tau_0) = Q_x(\tilde{x}_0(0) + \Theta\omega_0^{(0)}(\tau_0), \tilde{y}_0(0) + \Theta\nu_0^{(0)}(\tau_0))\omega_0^{(0)}(\tau_0)$$

$$+Q_y(\tilde{x}_0(0) + \Theta\omega_0^{(0)}(\tau_0), \tilde{y}_0(0) + \Theta\nu_0^{(0)}(\tau_0))\nu_0^{(0)}(\tau_0). \tag{2.10}$$

From the last equation and initial condition

$$\nu_0^{(0)}(0) = 0, \tag{2.11}$$

obtain that

$$\nu_0^{(0)}(\tau_0) = \exp\left( \int_0^{\tau_0} Q_y(t)dt \right) \int_0^{\tau_0} Q_x(s)\omega_0^{(0)}(s) \exp\left( -\int_0^{s} Q_y(t)dt \right) ds. \tag{2.12}$$

Taking into account condition $(C3)$, it follows that

$$|\nu_0^{(0)}(\tau_0)| \le c \exp(-\kappa\tau_0).$$

Assume that $x_0 < \varphi$. Then by means of (C3), (2.9), (2.10) and (2.11) it follows that

$$\dot{\nu}_0^{(0)}(\tau_0) > 0$$

for all $t$ near initial moment. However, if one can take into account the equality (2.12), this cannot be true for all $t > 0$. This is why the function $\nu_0^{(0)}(\tau_0)$ increases only to some moment of time $t = \xi(\varepsilon)$. The moment tends to the zero as $\varepsilon \to 0$ and the function $\dot{\nu}_0^{(0)}(\tau_0)$ is negative for $t > \xi$. Thus, there is a hump near the moment $t = \xi$.

To determine the coefficients of $\varepsilon^k$ for the approximation $\tilde{z}_k(t)$ and $\tilde{y}_k(t)$ we apply the system

$$\begin{cases} \tilde{x}'_{k-1}(t) = F_x(t)\tilde{x}_k(t) + F_y(t)\tilde{y}_k(0) + F_k(t), \\ \tilde{y}'_{k-1}(t) = Q_x(t)\tilde{x}_k(t) + Q_y(t)\tilde{y}_k(t) + Q_k(t). \end{cases}$$

To find $\omega_k^{(0)}(\tau_0)$ it is needed to solve the following system

$$\dot{\omega}_k^{(0)}(\tau_0) = F_x(\tau_0)\omega_k^{(0)}(\tau_0) + F_y(\tau_0)\nu_k^{(0)}(0) + G_k(\tau_0) = \Pi_k F(\tau_0),$$
$$\omega_k^{(0)}(0) = -\tilde{x}_k(0),\ \ \nu_k^{(0)}(0) = -\tilde{y}_k(0).$$

Hence, it is true

$$\omega_k^{(0)}(\tau_0) = -\tilde{x}_k(0)\exp\left(\int_0^{\tau_0} F_x(s)ds\right) + \int_0^{\tau_0}(-F_y(s)\tilde{y}_k(0) + G_k(s))\exp\left(\int_s^{\tau_0} F_x(t)dt\right)ds.$$

From the second equation of (2.8) and initial condition

$$\nu_k^{(0)}(0) = -\tilde{y}_k(0),$$

one can find

$$\nu_k^{(0)}(\tau_0) = -\tilde{y}_k(0)\exp\left(\int_0^{\tau_0} Q_y(s)ds\right) + \int_0^{\tau_0}\Big(Q_x(s)\omega_k^{(0)}(s) + T_k(s)\Big)\exp\left(\int_s^{\tau_0} Q_y(t)dt\right)ds.$$

By virtue of conditions $(C2)$ and $(C3)$, the following inequalities are hold,

$$\|\omega_k^{(0)}(\tau_0)\| \le c\exp(-\kappa\tau_0),\ \ \|\nu_k^{(0)}(\tau_0)\| \le c\exp(-\kappa\tau_0).$$

Consider the interval $t \in [\theta_i, \theta_{i+1}]$. For this interval, the initial values are

$$x(\theta_i,\varepsilon) = x(\theta_i), y(\theta_i,\varepsilon) = y(\theta_i). \tag{2.13}$$

Substituting the series (2.4) into initial values (2.13), one can obtain

$$\tilde{x}_0(\theta_i) + \varepsilon\tilde{x}_1(\theta_i) + .... + \omega_0^{(i)}(0) + \varepsilon\omega_1^{(i)}(0) + ... = x(\theta_i),$$
$$\tilde{y}_0(\theta_i) + \varepsilon\tilde{y}_1(\theta_i) + .... + \nu_0^{(i)}(0) + \varepsilon\nu_1^{(i)}(0) + ... = y(\theta_i).$$

Since the value $x(\theta_i,\varepsilon), y(\theta_i,\varepsilon)$ approach the point $(\varphi,\psi)$ as $\varepsilon \to 0$ [35], the following equalities hold,

$$\tilde{x}_0(\theta_i) = x(\theta_i),\ \ \tilde{x}_0(\theta_i) = y(\theta_i).$$

Hence, we equate the coefficients according to the powers of $\varepsilon$ in (2.13)

$$\omega_0^{(i)}(0) = 0, \tag{2.14}$$

$$\nu_0^{(i)}(0) = 0, \tag{2.15}$$

$$\omega_k^{(i)}(0) = -\tilde{x}_k(\theta_i),$$
$$\nu_k^{(i)}(0) = -\tilde{y}_k(\theta_i). \tag{2.16}$$

To find the coefficients of $\varepsilon^0$ for the approximation $\tilde{x}_0(t)$ and $\tilde{y}_0(t)$ we have the system

$$\begin{cases} 0 = F(\tilde{x}_0(t), \tilde{y}_0(\theta_i)), \\ 0 = Q(\tilde{x}_0(t), \tilde{y}_0(t)). \end{cases}$$

In view of conditions $(C2)$ and $(C3)$ we choose $\tilde{x}_0(t) = \tilde{x}(t) = \varphi(t), \tilde{y}_0(t) = \tilde{y}(t) = \psi(t)$. Using the mean value theorem and initial condition (2.15), the first equation of (2.8) one can write

$$\dot{\omega}_0^{(i)}(\tau_i) = F_x(\tilde{x}_0(\theta_i) + \Theta\omega_0^{(i)}(\tau_i), \tilde{y}_0(\theta_i))\omega_0^{(i)}(\tau_i),$$

where $0 < \Theta < 1$. From the last equation and initial condition (2.14) one can find

$$\omega_0^{(i)}(\tau_i) = 0. \tag{2.17}$$

The second equation of (2.8), in accordance with the mean value theorem and the equality (2.17), can be represented in the form

$$\dot{\nu}_0^{(i)}(\tau_i) = Q_y(\tilde{z}_0(\theta_i), \tilde{y}_0(\theta_i) + \Theta\nu_0^{(i)}(\tau_i))\nu_0^{(i)}(\tau_i),$$
$$\nu_0^{(i)}(0) = 0.$$

Therefore, one can obtain $\nu_0^{(i)}(\tau_i) = 0$. To determine the coefficients of $\varepsilon^k$ for the approximation $\tilde{z}_k(t)$ and $\tilde{y}_k(t)$ we apply the system

$$\begin{cases} \tilde{x}'_{k-1}(t) = F_x(t)\tilde{x}_k(t) + F_y(t)\tilde{y}_k(\theta_i) + F_k(t), \\ \tilde{y}'_{k-1}(t) = Q_x(t)\tilde{x}_k(t) + Q_y(t)\tilde{y}_k(t) + Q_k(t), \end{cases}$$

To find $\omega_k^{(0)}(\tau_0)$ it is needed to solve the following system

$$\dot{\omega}_k^{(i)}(\tau_i) = F_x(\tau_i)\omega_k^{(i)}(\tau_i) + F_y(\tau_i)\nu_k^{(i)}(0) + G_k(\tau_i),$$
$$\omega_k^{(i)}(0) = -\tilde{x}_k(\theta_i),$$
$$\nu_k^{(i)}(0) = -\tilde{y}_k(\theta_i).$$

Hence,

$$\omega_k^{(i)}(\tau_i) = -\tilde{x}_k(\theta_i)\exp\left(\int_0^{\tau_i} F_x(s)ds\right) + \int_0^{\tau_i} (-F_y(s)\tilde{y}_k(\theta_i) + G_k(s))\exp\left(\int_s^{\tau_i} F_x(t)dt\right) ds.$$

It remains to solve the equations

$$\dot{\nu}_k^{(i)}(\tau_i) = Q_x(\tau_i)\omega_k^{(i)}(\tau_i) + Q_y(\tau_i)\nu_k^{(i)}(\tau_i) + T_k(\tau_i).$$

By using condition (2.16) get that

$$\nu_k^{(i)}(\tau_i) = -\tilde{y}_k(\theta_i)\exp\left(\int_0^{\tau_i} Q_y(s)ds\right) + \int_0^{\tau_i} \left(Q_x(s)\omega_k^{(i)}(s) + T_k(s)\right)\exp\left(\int_s^{\tau_i} Q_y(t)dt\right) ds.$$

It can be proved that by virtue of conditions $(C2)$ and $(C3)$ the following inequalities hold,

$$|\omega_k^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), \quad |\nu_k^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), \quad i = 1, 2, ..., p.$$

Thus, the coefficients of the expansions (2.5) are obtained up to order $k = n$. In the asymptotic expansion of the solution the zeroth-order boundary-layer terms vanish at $t = \theta_i, i = 1, 2, ..., p$, i.e., $\nu_0^{(i)}(\tau_i) = 0$ and $\omega_0^{(i)}(\tau_i) = 0$. For all $k \ge 1$, the coefficients $\nu_k^{(i)}(\tau_i)$ and $\omega_k^{(i)}(\tau_i)$ are non-zero, so that a boundary layer is formally present in the expansion. However, its contribution is of order $\varepsilon^k$ with $k \ge 1$ and becomes negligibly small for sufficiently small $\varepsilon$. In practical applications, when constructing the solution, the influence of this boundary layer can therefore be safely neglected.

## 3. Main result

**Theorem 3.1.** *Let conditions $(C1)$ and $(C2)$ are hold. Then there exist positive constants $\varepsilon_0$ and $c$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the problem (2.2),(2.3) has a unique solution $x(t,\varepsilon), y(t,\varepsilon)$, which for $0 \le t \le T$ satisfies the inequalities,*

$$|x(t,\varepsilon) - X_n(t,\varepsilon)| \le c\varepsilon^{n+1}, \quad |y(t,\varepsilon) - Y_n(t,\varepsilon)| \le c\varepsilon^{n+1},$$

*where*

$$X_n(t,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \tilde{x}_k(t) + \sum_{k=0}^{n} \varepsilon^k \omega_k^{(i)}(\tau_i),$$
$$Y_n(t,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(t) + \sum_{k=0}^{n} \varepsilon^k \nu_k^{(i)}(\tau_i), \quad i = 1,2,...,p.$$

*Proof.* By replacing the variables $x(t,\varepsilon) = u(t,\varepsilon) + X_n(t,\varepsilon)$ and $y(t,\varepsilon) = v(t,\varepsilon) + Y_n(t,\varepsilon)$ in (2.2) and (2.3), we obtain the system

$$\text{(3.18)} \qquad \begin{cases} \varepsilon \dfrac{du(t,\varepsilon)}{dt} = F_x u(t,\varepsilon) + F_y v(\beta(t),\varepsilon) + G_1(u(t,\varepsilon), v(\beta(t),\varepsilon), t, \varepsilon), \\ \varepsilon \dfrac{dv(t,\varepsilon)}{dt} = Q_x u(t,\varepsilon) + Q_y v(t,\varepsilon) + G_2(u(t,\varepsilon), v(t,\varepsilon), t, \varepsilon), \end{cases}$$

with initial condition

$$\text{(3.19)} \qquad u(0,\varepsilon) = 0, \quad v(0,\varepsilon) = 0,$$

where the elements of matrices $F_x, F_y, Q_x$ and $Q_y$ are calculated at the points $(\tilde{z}_0(t) + \omega_0^{(i)}(\tau_i), \tilde{y}_0(t), 0), i = 1,2,...,p,$

$$\begin{aligned} G_1(u(t,\varepsilon), v(\beta(t),\varepsilon), t, \varepsilon) &= F(u(t,\varepsilon) + X_n(t,\varepsilon), v(\beta(t),\varepsilon) + Y_n(\beta(t),\varepsilon)) \\ &\quad - \varepsilon \frac{dX_n(t,\varepsilon)}{dt} - F_x u(t,\varepsilon) - F_y v(\beta(t),\varepsilon), \\ G_2(u(t,\varepsilon), v(t,\varepsilon), t, \varepsilon) &= f(u(t,\varepsilon) + X_n(t,\varepsilon), v(t,\varepsilon) + Y_n(t,\varepsilon)) \\ &\quad - \varepsilon \frac{dY_n(t,\varepsilon)}{dt} - Q_x u(t,\varepsilon) - Q_y v(t,\varepsilon). \end{aligned}$$

The functions $G(u,v,t,\varepsilon)$ have the following two properties:
1) $G_1(0,0,t,\varepsilon) = O(\varepsilon^{n+1}), G_2(0,0,t,\varepsilon) = O(\varepsilon^{n+1})$.
2) For any $\varepsilon > 0$ there exist numbers $\delta = \delta(\varepsilon)$ and $\mu = \mu(\varepsilon)$ such that for $\|u_i\| \le \delta$, $\|v_i\| \le \delta, \ i = 1,2, \ 0 < \varepsilon < \varepsilon_0$ the following inequalities hold,
$\|G_i(u_1,v_1,t,\varepsilon) - G_i(u_2,v_2,t,\varepsilon)\| \le \varepsilon(\|u_2 - u_1\| + \|v_2 - v_1\|), \ \ i = 1,2.$

Let us prove the property 1). For $t \in (\theta_i, \theta_{i+1}]$ we obtain that

$$\begin{aligned} G_1(0,0,t,\varepsilon) &= F(X_n(t,\varepsilon), Y_n(\beta(t),\varepsilon)) - \varepsilon \frac{dX_n}{dt} \\ &= \Bigg[ F\bigg( \sum_{k=0}^{n} \varepsilon^k (\tilde{x}_k(\varepsilon\tau_i) + \omega_k^{(i)}(\tau_i)), \sum_{k=0}^{n} \varepsilon^k (\tilde{y}_k(\theta_i) + \varepsilon \nu_k^{(i)}(0)) \bigg) \\ &\quad - F\bigg( \sum_{k=0}^{n} \varepsilon^k \tilde{x}_k(\varepsilon\tau_i), \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(\theta_i) \bigg) - \sum_{k=0}^{n} \varepsilon^k \dot{\omega}_k^{(i)}(\tau_i) \Bigg] \\ &\quad + \Bigg[ F\bigg( \sum_{k=0}^{n} \varepsilon^k \tilde{x}_k(t), \sum_{k=0}^{n} \varepsilon^k \tilde{y}_k(\theta_i) \bigg) - \varepsilon \sum_{k=0}^{n} \varepsilon^k \tilde{x}_k'(t) \Bigg] \\ &= \Bigg[ \sum_{k=0}^{n} \varepsilon^k \Pi_k F(\tau_i) + O(\varepsilon^{n+1}) - \sum_{k=0}^{n} \varepsilon^k \dot{\omega}_k^{(i)}(\tau_i) \Bigg] \\ &\quad + \Bigg[ \sum_{k=0}^{n} \varepsilon^k \tilde{F}_k(t) + O(\varepsilon^{n+1}) - \sum_{k=0}^{n} \varepsilon^k \tilde{x}_k'(t) \Bigg] = O(\varepsilon^{n+1}), \end{aligned}$$

similarly to that for functions $\tilde{y}_k(t), \nu_k^{(i)}(\tau_i), i = 1, 2, ..., p$. The second property of the functions $G_j, j = 1, 2$, follows the mean value theorem. Actually,

$$G_i(u_1, v_1, t, \varepsilon) - G_i(u_2, v_2, t, \varepsilon) = \int_0^1 \frac{\partial}{\partial s} G_i(su_1 + (1-s)u_2, sv_1 + (1-s)v_2, t, \varepsilon)ds$$

$$= \int_0^1 \partial_u G_i(u^*(s), v^*(s), t, \varepsilon)ds \cdot (u_1 - u_2) + \int_0^1 \partial_v G_i(u^*(s), v^*(s), t, \varepsilon)ds \cdot (v_1 - v_2),$$

where $u^*(s) = su_1 + (1-s)u_2,\ v^*(s) = sv_1 + (1-s)v_2$.

Moreover,

$$\partial_u G_i(u^*(s), v^*(s), t, \varepsilon) = \partial_x F(u^*(s) + X_n, v^*(s) + Y_n, \varepsilon) - \partial_x F(X_0, Y_0, 0),$$
$$\partial_v G_i(u^*(s), v^*(s), t, \varepsilon) = \partial_y F(u^*(s) + Z_n, v^*(s) + Y_n, \varepsilon) - \partial_y F(Z_0, Y_0, 0),$$

and

$$|u^*(s) + X_n(t, \varepsilon) - X_0(t)| \le |u^*(s)| + C\varepsilon,$$
$$|v^*(s) + Y_n(t, \varepsilon) - Y_0(t)| \le |v^*(s)| + C\varepsilon.$$

The continuity of the first derivatives of the functions $F(z, y, \varepsilon)$ and $f(z, y)$ confirms the validity of property 2).

Now, we replace the system (3.18), (3.19) by the equivalent integral equations

$$u(t, \varepsilon) = \frac{1}{\varepsilon} \int_0^t U(t, s, \varepsilon)[F_y(s, \varepsilon)v(\beta(s), \varepsilon) + G_1(u(s, \varepsilon), v(\beta(s), \varepsilon), s, \varepsilon)]ds, \tag{3.20}$$

$$v(t, \varepsilon) = \frac{1}{\varepsilon} \int_0^t V(t, s, \varepsilon)[Q_x(s, \varepsilon)u(s, \varepsilon) + G_2(u(s, \varepsilon), v(s, \varepsilon), s, \varepsilon)]ds, \tag{3.21}$$

where $U(t, s, \varepsilon)$ and $V(t, s, \varepsilon)$ are the fundamental matrices of the system

$$\begin{cases} \varepsilon \dfrac{dU}{dt} = F_x(t, \varepsilon)U, \quad t \ne \theta_i, \quad U(s, s, \varepsilon) = E, \\ \varepsilon \dfrac{dV}{dt} = Q_y(t, \varepsilon)V, \quad t \ne \theta_i, \quad V(s, s, \varepsilon) = E. \end{cases}$$

For the matrix $U(t, s, \varepsilon)$ it is correct that

$$\|U(t, s, \varepsilon)\| \le c\exp(-\frac{\kappa}{\varepsilon}(t - s)), \qquad \|V(t, s, \varepsilon)\| \le c\exp(-\frac{\kappa}{\varepsilon}(t - s)), \ \ 0 \le s \le t \le T.$$

Substituting the expression for $v(t, \varepsilon)$, defined by the equation (3.21), into the equation (3.20), we obtain

$$u(t, \varepsilon) = \int_0^t K(t, s, \varepsilon)u(s, \varepsilon)ds + W_1(u(t, \varepsilon), v(\beta(t), \varepsilon), t, \varepsilon),$$

where $K$ is a bounded kernel, and the operator $W_1$ possesses the same two properties as function $G(u, v, t, \varepsilon)$. The last equation can be replaced by the equivalent equation

$$\begin{aligned} u(t, \varepsilon) &= \int_0^t R(t, s, \varepsilon)W_1(u(s, \varepsilon), v(\beta(s), \varepsilon), s, \varepsilon)ds + W_1(u(t, \varepsilon), v(\beta(t), \varepsilon), t, \varepsilon) \\ &= T_1(u(t, \varepsilon), v(\beta(t), \varepsilon), t, \varepsilon), \end{aligned} \tag{3.22}$$

where $R$ is the resolvent of the kernel $K$. Now, let us substitute the expression (3.22) for $u(t,\varepsilon)$ into the equation of (3.21),

(3.23)
$$v(t,\varepsilon) = \int_0^t V(t,s,\varepsilon)[Q_x(s,\varepsilon)T_1(u(s,\varepsilon),v(\beta(s),\varepsilon),t,\varepsilon) + G_2(u(s,\varepsilon),v(s,\varepsilon),t,\varepsilon)]ds$$
$$= T_2(u(t,\varepsilon),v(\beta(t),\varepsilon),t,\varepsilon).$$

Integral operators $T_1$ and $T_2$ admit the same two properties as function $G(u,v,t,\varepsilon)$. Applying the method of successive approximations to the systems (3.22), (3.23) we find that a unique solution exists and satisfies the estimates

$$|u(t,\varepsilon)| = |x(t,\varepsilon) - X_n(t,\varepsilon)| \le c\varepsilon^{n+1},\ |v(t,\varepsilon)| = |y(t,\varepsilon) - Y_n(t,\varepsilon)| \le c\varepsilon^{n+1},\ \ 0 \le t \le T.$$

The theorem is proved. □

## 4. Numerical examples

Consider the system with singularities

$$\begin{cases} \varepsilon x' = 6 - xy(\beta(t)), \\ \varepsilon y' = 13 - x^2 - y^2, \end{cases} \tag{4.24}$$

and initial conditions

$$x(0,\varepsilon) = 6,\ y(0,\varepsilon) = 3, \tag{4.25}$$

where $\theta_i = i/4, i = 1,2,...8, \varphi = 2, \psi = 3$. Let us now verify the conditions of theorem 3.1. The system (4.24) is of the form (2.2) with $F(x,y) = 6 - xy(\beta(t))$, $Q(x,y) = 13 - x^2 - y^2$, $F(2,3) = 0$, $Q(2,3) = 0$, $F_x(2,3) = -3 < 0$, $F_y(2,3) = -2 < 0$, $Q_x(2,3) = -6 < 0$, $Q_y(2,3) = -4 < 0$. The presence of a hump in the boundary layer at $t = 0$ and the absence of a boundary layer at the points $t = \theta_i$, $i = 1,2,\dots,p$, are illustrated in Figure 1. This behavior is consistent with the theoretical result that the zeroth-order boundary-layer terms vanish at each point $t = \theta_i$, i.e., $\nu_0^{(i)}(\tau_i) = 0$ and $\omega_0^{(i)}(\tau_i) = 0$. Therefore, no boundary layer appears at these points.Figure 2 demonstrates the solution of system (4.24) with initial value (4.25) which tends to $x = \varphi = 2$ as $\varepsilon \to 0$.

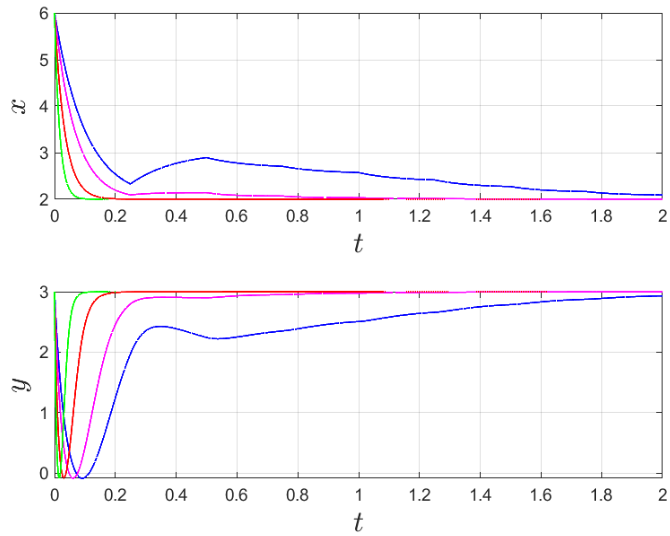


FIGURE 1. The red, blue and green lines are graphs of solutions of system (4.24) with initial values $x(0,\varepsilon) = 6$ and $y(0,\varepsilon) = 3$ with values of $\varepsilon$ : $0.3, 0.2, 0.1, 0.05$, respectively.

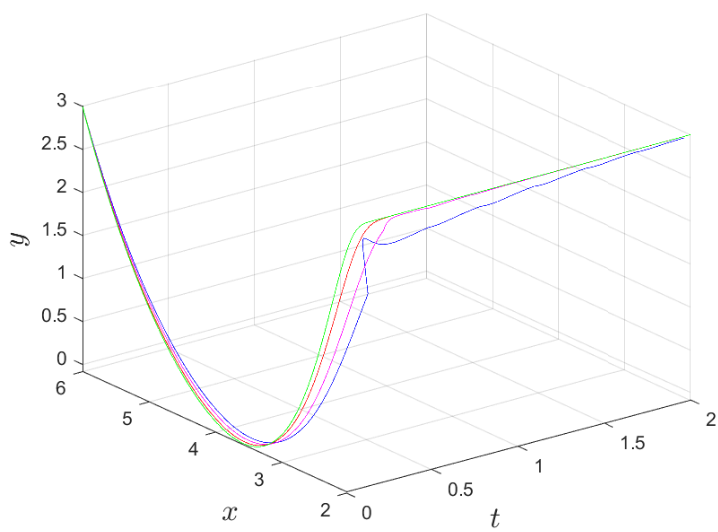


FIGURE 2. The red, blue and green lines are graphs of solutions of system (4.24) with initial values $x(0,\varepsilon) = 6$ and $y(0,\varepsilon) = 3$ with values of $\varepsilon$ : $0.3, 0.2, 0.1, 0.05$, respectively.

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