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On the Split Combination Feasibility Problem with an Application to Bandwidth Allocation

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ABSTRACT. We introduce the *split combination feasibility problem* (SCFP), an extension of the classical split feasibility problem that mixes two bounded linear operators A and B via convex combinations. We provide sufficient conditions for existence and uniqueness of solutions and establish strong convergence of a projection-based iterative algorithm. The framework unifies and extends known approaches and accommodates key applications, including the combination of variational inequalities and constrained convex minimization. Practical relevance is illustrated by a bandwidth-allocation example under mixed routing, where the proposed scheme converges strongly to a feasible rate vector. These results demonstrate that the SCFP offers a robust and versatile lens for modeling feasibility with operator mixing, while delivering provable convergence guarantees and unique solutions under suitable assumptions.

1. INTRODUCTION

Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators.

The *classical split feasibility problem* (SFP), introduced by Censor and Elfving [2], is to find a point $x \in C$ such that $Ax \in Q$. The set of all solutions is denoted by

$$\Gamma_A = \{\, x \in C \mid Ax \in Q \,\}.$$

Motivated by the flexibility of the split framework and its numerous applications (see, e.g., [1, 3, 4]), we introduce in this paper a new problem, called the *split combination feasibility problem* (SCFP). The SCFP is to find $x \in C$ such that

$$aAx + (1-a)Bx \in Q, \qquad \forall a \in [0,1].$$

The solution set of the SCFP is denoted by

$$\Gamma^a_{A,B} = \{\, x \in C \mid aAx + (1-a)Bx \in Q,\ \forall a \in [0,1] \,\}.$$

This model extends the classical SFP by incorporating a convex combination of two bounded linear operators, thus providing a broader framework that can capture more complex feasibility structures arising in optimization and applied problems.

To illustrate the structure of the newly introduced split combination feasibility problem, we provide the following examples.

Example 1.1. *Let $H = L^2([0,1])$ and set $H_1 = H_2 = H$. Define the closed convex set*

$$Q := \big\{\, y \in L^2([0,1]) \;\big|\; y(t) = 0 \text{ a.e. on } (1/2, 1] \,\big\},$$

and take $C := H$.

Let $A = I$ (the identity on H) and let $B = M_t$ be the multiplication operator

$$(Bx)(t) := t\,x(t), \qquad t \in [0,1].$$

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Then A and B are bounded linear operators with $\|A\| = 1$ and $\|B\| = \|t\|_\infty = 1$.

For $a \in [0,1]$ and $x \in H$ we have

$$\big(aA + (1-a)B\big)x \;=\; a\,x + (1-a)\,t\,x.$$

If x is supported in $[0,1/2]$ (i.e., $x \in Q$), then $t\,x$ is also supported in $[0,1/2]$; hence $a\,x + (1-a)\,t\,x \in Q$ for every $a \in [0,1]$. Therefore

$$Q \subseteq \Gamma^{a}_{A,B}.$$

Conversely, if $x \in \Gamma^{a}_{A,B}$, then for all $a \in [0,1]$ we have

$$(M_a x)(t) = \big(a + (1-a)t\big)x(t).$$

Since $M_a x \in Q$, it follows that $(M_a x)(t) = 0$ a.e. on $(1/2,1]$. On this interval the coefficient satisfies

$$a + (1-a)t \;\geq\; \tfrac{a+1}{2} \;>\; 0,$$

so necessarily $x(t) = 0$ a.e. on $(1/2,1]$, that is, $x \in Q$.

Hence

$$\Gamma^{a}_{A,B} = Q = \big\{\, x \in L^2([0,1]) \;\big|\; x(t) = 0 \text{ a.e. on } (1/2,1] \,\big\}.$$

In particular, any x supported in $[0,1/2]$ is a solution, and the entire solution set is precisely the closed convex set Q.

Example 1.2. *Let $H_1 = \mathbb{R}^2$, $H_2 = \mathbb{R}^3$, $C = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$, and $Q = \{y \in \mathbb{R}^3 : y_1, y_2, y_3 \geq 0\}$. Define*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For $x = (x_1, x_2) \in C$ and $a \in [0,1]$ we have

$$aAx + (1-a)Bx = (ax_1, (1-a)x_2, 0).$$

Since Q is convex, the requirement $aAx + (1-a)Bx \in Q$ for all $a \in [0,1]$ is equivalent to $Ax \in Q$ and $Bx \in Q$, i.e. $x_1 \geq 0$ and $x_2 \geq 0$. Hence

$$\Gamma^{a}_{A,B} = \big\{(x_1, x_2) \in \mathbb{R}^2 :\; x_1 \geq 0,\; x_2 \geq 0,\; x_1^2 + x_2^2 \leq 1\big\}.$$

These examples demonstrate that the solution set $\Gamma^{a}_{A,B}$ can vary significantly in form, sometimes coinciding with a large convex set and sometimes admitting a more restricted description. This observation motivates a deeper analysis of conditions under which $\Gamma^{a}_{A,B}$ reduces to a singleton, which is one of the central contributions of the present paper.

Remark 1.1. From Example 1.2, we observe that the solution set

$$\Gamma^{a}_{A,B}$$

may contain multiple elements, which makes it difficult to apply directly in practical settings. To overcome this difficulty, we shall employ Lemma 2.7 in Section 2, which provides sufficient conditions to ensure that $\Gamma^{a}_{A,B}$ becomes a singleton. Having a unique solution greatly simplifies both theoretical analysis and practical implementation in real-world problems, such as *bandwidth allocation in communication networks*, *traffic equilibrium models*, and *image reconstruction problems*.

Bandwidth allocation and related works. The problem of bandwidth allocation has been extensively studied within the framework of *network utility maximization* (NUM). In this classical setting, the aim is to maximize the aggregate utility of all sources subject to link capacity constraints [8, 10]. A large body of work has investigated distributed algorithms for NUM, including projected gradient, Newton-type, and subgradient schemes; see, for example, Iiduka [6] and references therein. Such methods are particularly relevant when utility functions are nonsmooth or when only local capacity information is available at each link.

Another important line of research concerns *bandwidth-sharing networks with rate constraints*, as studied by Frolkova, Reed, and Zwart [5]. Using fluid and diffusion approximations, they established the existence and uniqueness of invariant points for large-scale communication networks and analyzed the stability and performance of flow-level models. These results provide deep theoretical insights but rely on stochastic and asymptotic assumptions at the network level.

In contrast, the present paper introduces a novel perspective by formulating bandwidth allocation within the framework of the *split combination feasibility problem* (SCFP). Here, two routing operators A and B are combined through convex mixing, leading to the feasibility set

$$\Gamma^{a}_{A,B} = \{\, x \in C : aAx + (1-a)Bx \in Q,\ \forall a \in [0,1] \,\}.$$

This setting allows us to capture robustness with respect to mixed routing patterns, which is not explicitly addressed in the above-mentioned works. To illustrate the applicability of our approach, we provide a concrete numerical example of bandwidth allocation under mixed routing, showing that the iterative scheme developed in this paper converges strongly to a feasible rate vector. This example highlights a key distinction of our contribution: rather than focusing only on stochastic approximations or classical NUM, we demonstrate that the SCFP framework can ensure convergence and uniqueness of feasible allocations in a deterministic Hilbert space setting.

In summary, this paper introduces and studies the *split combination feasibility problem* (SCFP), a new generalization of the classical split feasibility problem. We establish theoretical results that guarantee the existence and uniqueness of solutions under suitable conditions, and we develop an iterative algorithm whose strong convergence is proved by our main theorem. The framework is then applied to two important classes of problems: the combination of variational inequalities and constrained convex minimization. Finally, to demonstrate the practical relevance of our approach, we present a detailed numerical example on bandwidth allocation under mixed routing, where the proposed algorithm is shown to converge effectively to a feasible rate vector. The remainder of the paper is organized as follows: Section 2 provides preliminaries and auxiliary lemmas, Section 3 presents the main convergence theorem, and Section 4 discusses applications to variational inequalities, convex minimization, and bandwidth allocation.

2. Preliminaries and Related Theorems

In this section, we recall some definitions, properties, and auxiliary results which will be used in proving our main theorems.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. For a nonempty closed convex subset $C \subseteq H$, the *metric projection* $P_C : H \to C$ is defined by

$$P_C(x) = \arg\min_{y \in C} \|x - y\|, \qquad x \in H.$$

It is well known that P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y,\, x - y\rangle, \qquad \forall x, y \in H.$$

Definition 2.1. A mapping $T : C \to C$ is said to be

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(ii) *α-contractive*, $\alpha \in (0, 1)$, if $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$;
(iii) *η-inverse strongly monotone*, $\eta > 0$, if

$$\langle Tx - Ty,\, x - y\rangle \geq \eta\|Tx - Ty\|^2, \qquad \forall x, y \in C.$$

The fixed point problem of $T : C \to C$ is to find $x^* \in C$ such that $Tx^* = x^*$. The set of all fixed points of T is denoted by $F(T)$.

Definition 2.2. Let $C \subseteq H$ be nonempty closed convex and let $G : C \to H$. The *variational inequality problem* (VIP) is to find $x^* \in C$ such that

$$\langle G(x^*),\, y - x^*\rangle \geq 0, \qquad \forall y \in C.$$

The set of all solutions is denoted by $VI(C, G)$.

We now list several useful lemmas.

Lemma 2.3 ([11]). *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if*

$$\langle x - y,\, y - z\rangle \geq 0, \qquad \forall z \in C.$$

Lemma 2.4 ([12]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $A : C \to H$ a mapping. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C, A) \quad \Longleftrightarrow \quad u = P_C(I - \lambda A)u.$$

Lemma 2.5 ([13]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \qquad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{\delta_n\}$ is a sequence such that either $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6 ([9]). *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}^+$, $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n, \qquad \forall n \geq 0.$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(a) If $b_n \leq \alpha_n C$ where $C \geq 0$, then $\{a_n\}$ is a bounded sequence.
(b) If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \dfrac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

These lemmas will serve as fundamental tools for establishing the convergence results in the subsequent sections.

Lemma 2.7. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with adjoints A^* and B^*. Assume that $\Gamma^{a}_{A,B} \neq \varnothing$ for all $a \in [0, 1]$.*

Let L_A and L_B denote the spectral radius of A^*A and B^*B, respectively. Suppose $\lambda \in (0, \frac{2}{L})$ with $L = \max\{L_A, L_B\}$.

Then the following statements are equivalent:

(i) $x^* \in \Gamma^{a}_{A,B}$,
(ii) $x^* = P_C\big[(I - \lambda M^*(I - P_Q)M)x^*\big]$,

where $M = aA + (1-a)B$ and M^* is the adjoint of M.

Proof. Suppose that the conditions are satisfied. Since A^* and B^* are the adjoints of A and B, and

$$M = aA + (1-a)B,$$

it follows that

$$M^* = aA^* + (1-a)B^*$$

is the adjoint of M.

(i) $\Rightarrow$ (ii): Suppose $x^* \in \Gamma^a_{A,B}$. Then we have $x^* \in C$ and

$$Mx^* = aAx^* + (1-a)Bx^* \in Q.$$

Therefore, it follows that

$$x^* = P_C x^* \quad \text{and} \quad Mx^* = P_Q M x^*.$$

Hence,

$$x^* = P_C\big[(I - \lambda M^*(I - P_Q)M)x^*\big].$$

(ii) $\Rightarrow$ (i): Suppose

$$x^* = P_C\big[(I - \lambda M^*(I - P_Q)M)x^*\big].$$

By the property of P_C, it follows that $x^* \in C$.

Let $w \in \Gamma^a_{A,B}$. According to (i), we obtain

$$w = P_C\big[(I - \lambda M^*(I - P_Q)M)w\big].$$

Given that $I - P_Q$ is firmly nonexpansive and $\lambda \in \big(0, \frac{2}{L}\big)$, the following inequality holds:

$$\begin{aligned}
\|x^* - w\|^2 &\le \big\|x^* - w - \lambda M^*\big((I - P_Q)Mx^* - (I - P_Q)Mw\big)\big\|^2 \\
&= \|x^* - w\|^2 - 2\lambda\big\langle (I - P_Q)Mx^*, Mx^* - Mw\big\rangle \\
&\quad + \lambda^2\big\|(aA^* + (1-a)B^*)(I - P_Q)Mx^*\big\|^2 \\
&\le \|x^* - w\|^2 - 2\lambda\|(I - P_Q)Mx^*\|^2 \\
&\quad + \lambda^2 a\|A^*(I - P_Q)Mx^*\|^2 + \lambda^2(1-a)\|B^*(I - P_Q)Mx^*\|^2 \\
&\le \|x^* - w\|^2 - 2\lambda\|(I - P_Q)Mx^*\|^2 \\
&\quad + \lambda^2 aL\|(I - P_Q)Mx^*\|^2 + \lambda^2(1-a)L\|(I - P_Q)Mx^*\|^2 \\
&= \|x^* - w\|^2 - 2\lambda\|(I - P_Q)Mx^*\|^2 + \lambda^2 L\|(I - P_Q)Mx^*\|^2 \\
&= \|x^* - w\|^2 - \lambda(2 - \lambda L)\,\|(I - P_Q)Mx^*\|^2.
\end{aligned}$$

Therefore, we have

$$Mx^* = P_Q M x^*.$$

Consequently,

$$aAx^* + (1-a)Bx^* \in Q.$$

Thus, it follows that $x^* \in \Gamma^a_{A,B}$.

$\square$

Remark 2.8. It follows directly from Lemma 2.7 that the mapping

$$P_C\big[(I - \lambda M^*(I - P_Q)M)w\big]$$

can be characterized as a nonexpansive mapping.

Example 2.1. *Let $H_1 = H_2 = \mathbb{R}^2$,*

$$C = \{x \in \mathbb{R}^2 : \|x\|_2 \le 1\}, \qquad Q = \{(y_1, y_2) \in \mathbb{R}^2 : \ y_1 = y_2\}.$$

Define the bounded linear operators

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $x = (x_1, x_2) \in C$ and $a \in [0, 1]$,

$$\big(aA + (1-a)B\big)x = (a\,x_1,\ (1-a)\,x_2).$$

If $x \in \Gamma^a_{A,B}$, then $(ax_1, (1-a)x_2) \in Q$ for all $a \in [0,1]$. Since $Q = \{(y_1, y_2) : y_1 = y_2\}$, this requires

$$ax_1 = (1-a)x_2 \quad \textit{for all } a \in [0,1].$$

Taking $a = 1$ gives $x_1 = 0$, and taking $a = 0$ gives $x_2 = 0$, so $x = (0, 0)$. Conversely, $(0, 0) \in C$ and clearly $(aA + (1-a)B)(0, 0) = (0, 0) \in Q$ for all $a \in [0, 1]$. Hence

$$\Gamma^a_{A,B} = \{(0, 0)\}.$$

This shows that the solution set reduces to the singleton $\{(0, 0)\}$. Hence, although Q has many points, the solution set consists of only one point. Moreover,

$$A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^*B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ \Rightarrow\ L_A = L_B = 1,\ \ L = \max\{L_A, L_B\} = 1,$$

so any $\lambda \in (0, 2)$ satisfies the step-size requirement in Lemma 2.7.

To see this more explicitly, recall that

$$P_C(z) = \begin{cases} z, & \|z\|_2 \le 1, \\ \dfrac{z}{\|z\|_2}, & \|z\|_2 > 1, \end{cases} \qquad P_Q(y_1, y_2) = \Big(\tfrac{y_1+y_2}{2},\ \tfrac{y_1+y_2}{2}\Big).$$

For $x^* = (0, 0)$ we have

$$Mx^* = (0, 0), \quad (I - P_Q)Mx^* = (0, 0), \quad M^*(I - P_Q)Mx^* = (0, 0),$$

and hence

$$(I - \lambda M^*(I - P_Q)M)x^* = (0, 0).$$

Since $\|x^*\| = 0 \le 1$, it follows that

$$P_C\big[(I - \lambda M^*(I - P_Q)M)x^*\big] = P_C(0, 0) = (0, 0) = x^*.$$

Therefore, condition (ii) *of Lemma 2.7 is satisfied, and the equivalence in the lemma holds for this example.*

3. Main result

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with adjoints A^*, B^*. Let L_A and L_B be the spectral radius of A^*A and B^*B, respectively, and set*

$$L = \max\{L_A, L_B\}.$$

Assume that $\Gamma^a_{A,B} \cap F(T) \neq \varnothing$ for all $a \in [0, 1]$. Let $T : C \to C$ be nonexpansive, and let $\{x_n\}$ be the sequence generated from $x_1 \in C$ by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T x_n + \gamma_n P_C(I - \lambda M^*(I - P_Q)M)x_n, \qquad n \ge 1,$$

where $M = aA + (1-a)B$, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $f : C \to C$ is an α-contractive mapping for some $\alpha \in (0, 1)$.

Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) *there exist* $c, d > 0$ *such that* $c \le \beta_n, \gamma_n \le d$ *for all* n;

(iii) $\lambda \in \left(0, \frac{2}{L}\right)$;

(iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then $\{x_n\}$ *converges strongly to the point*

$$x_0 = P_{F(T)\cap\Gamma^a_{A,B}} f(x_0).$$

Proof. Let $v \in F(T)\cap\Gamma^a_{A,B}$. From Remark 2.8 and the recursive formula of $\{x_n\}$, we obtain

$$\begin{aligned}\|x_{n+1} - v\| &\le \alpha_n\|f(x_n) - v\| + \beta_n\|Tx_n - v\| + \gamma_n\big\|P_C\big[(I - \lambda M^*(I - P_Q)M)x_n\big] - v\big\| \\ &\le \alpha_n\|f(x_n) - f(v)\| + \alpha_n\|f(v) - v\| + (1 - \alpha_n)\|x_n - v\| \\ &\le (1 - \alpha_n(1 - \alpha))\|x_n - v\| + \alpha_n(1 - \alpha)\frac{\|f(v) - v\|}{1 - \alpha}.\end{aligned}$$

From Lemma 2.6, the sequence $\{x_n\}$ is bounded, and consequently $\{Tx_n\}$ and $\{P_C(I - \lambda M^*(I - P_Q)M)x_n\}$ are also bounded.

Now set

$$D = P_C\big(I - \lambda M^*(I - P_Q)M\big).$$

From the recursive formula of $\{x_n\}$, we then have

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\,\|f(x_{n-1})\| \\ &\quad + \beta_n\|Tx_n - Tx_{n-1}\| + |\beta_n - \beta_{n-1}|\,\|Tx_{n-1}\| \\ &\quad + \gamma_n\|Dx_n - Dx_{n-1}\| + |\gamma_n - \gamma_{n-1}|\,\|Dx_{n-1}\| \\ &\le (1 - \alpha_n(1 - \alpha))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\,\|f(x_{n-1})\| \\ &\quad + |\beta_n - \beta_{n-1}|\,\|Tx_{n-1}\| + |\gamma_n - \gamma_{n-1}|\,\|Dx_{n-1}\|.\end{aligned}$$

From Lemma 2.5, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.1}$$

From the recursive formula of $\{x_n\}$, we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|Tx_n - v\|^2 + \gamma_n\|Dx_n - v\|^2 \\ &\quad - \beta_n\gamma_n\|Tx_n - Dx_n\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|x_n - v\|^2 - \beta_n\gamma_n\|Tx_n - Dx_n\|^2.\end{aligned}$$

It implies that

$$\beta_n\gamma_n\|Tx_n - Dx_n\|^2 \le \alpha_n\|f(x_n) - v\|^2 + (\|x_{n+1} - v\| + \|x_n - v\|)\|x_{n+1} - x_n\|. \tag{3.2}$$

From condition (ii) and (3.2), we obtain

$$\lim_{n\to\infty} \|Tx_n - Dx_n\| = 0. \tag{3.3}$$

From the recursive formula of $\{x_n\}$, we also have

$$x_{n+1} - Tx_n = \alpha_n\big(f(x_n) - Tx_n\big) + \gamma_n(Dx_n - Tx_n), \tag{3.4}$$

and

$$x_{n+1} - Dx_n = \alpha_n\big(f(x_n) - Dx_n\big) + \beta_n(Tx_n - Dx_n). \tag{3.5}$$

From condition (i), together with (3.3), (3.4) and (3.5), it follows that

$$\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = \lim_{n\to\infty} \|x_{n+1} - Dx_n\| = 0. \tag{3.6}$$

Finally, from (3.1) and (3.6), we deduce that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Dx_n\| = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $q \in C$.

Suppose that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty} \langle f(x^*) - x^*, x_{n_k} - x^*\rangle, \tag{3.8}$$

where $x^* = P_{F(T)\cap\Gamma^a_{A,B}} f(x^*)$.

Assume that $q \notin F(T)$. From (3.7) and Opial's condition, we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \|x_{n_k} - q\| &< \limsup_{k\to\infty} \|x_{n_k} - Tq\| \\
&\le \limsup_{k\to\infty} \big[\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tq\|\big] \\
&\le \limsup_{k\to\infty} \|x_{n_k} - q\|.
\end{aligned}$$

This is a contradiction. Hence, $q \in F(T)$.

By a similar argument, we can also conclude that

$$q \in F\big(P_C(I - \lambda M^*(I - P_Q)M)\big).$$

From Lemma 2.7 , it then follows that

$$q \in \Gamma^a_{A,B}.$$

Therefore, we conclude that

$$q \in F(T) \cap \Gamma^a_{A,B}.$$

From (3.8), we have

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty} \langle f(x^*) - x^*, x_{n_k} - x^*\rangle = \langle f(x^*) - x^*, q - x^*\rangle \le 0, \tag{3.9}$$

where $x^* = P_{F(T)\cap\Gamma^a_{A,B}} f(x^*)$.

From the recursive formula of $\{x_n\}$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \big\|\alpha_n(f(x_n) - x^*) + \beta_n(Tx_n - x^*) + \gamma_n(Dx_n - x^*)\big\|^2 \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\alpha\|x_n - x^*\|\,\|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\quad + \alpha_n\alpha\|x_{n+1} - x^*\|^2 + \alpha_n\alpha\|x_n - x^*\|^2.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{}& \left(1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}\right)\|x_n - x^*\|^2 \\
&+ \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}\left(\frac{\alpha_n}{2(1-\alpha)}\|x_n - x^*\|^2 + \frac{1}{1-\alpha}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\right).
\end{aligned}$$

From condition (i), (3.9), and Lemma 2.5, we conclude that the sequence $\{x_n\}$ converges strongly to

$$x^* = P_{F(T)\cap\Gamma^a_{A,B}} f(x^*).$$

This completes the proof.

□

To further clarify the relationship between the proposed split combination feasibility problem and the classical split feasibility problem, we present the following remark.

Remark 3.2. If $A = B$, then the split combination feasibility problem reduces to the classical split feasibility problem. In this case, we have $\Gamma^a_{A,B} = \Gamma_A$ for all $a \in [0,1]$, and hence Theorem 3.1 coincides with the standard convergence result for the split feasibility problem.

To illustrate the feasibility of the assumptions in Theorem 3.1, we present the following example, which is constructed based on the setting of Example 2.1. This example shows that all hypotheses of Theorem 3.1 can be satisfied simultaneously in a concrete and simple framework.

Example 3.1. *Let $H_1 = H_2 = \mathbb{R}^2$ with the usual inner product and norm. Let*

$$C = \{x \in \mathbb{R}^2 : \|x\| \le 1\}, \qquad Q = \{(y_1, y_2) \in \mathbb{R}^2 :\ y_1 = y_2\}.$$

Define bounded linear operators

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $a \in [0,1]$, set $M = aA + (1-a)B$. As shown in Example 2.1, the associated split combination feasibility set is

$$\Gamma^a_{A,B} = \{(0,0)\}.$$

Let

$$S = \{(t,0) \in \mathbb{R}^2 :\ |t| \le 1\} \subset C,$$

and define $T : C \to C$ by $T = P_S$. Let $f : C \to C$ be defined by $f(x) = \frac{1}{2}x$.
Since $(0,0) \in S$, it follows that

$$\Gamma^a_{A,B} \cap F(T) \neq \varnothing.$$

Moreover,

$$L_A = 1, \qquad L_B = 1, \qquad L = \max\{L_A, L_B\} = 1,$$

and we choose $\lambda = \frac{1}{2} \in (0, 2/L)$.
Let

$$\alpha_n = \frac{1}{n+1}, \qquad \beta_n = \gamma_n = \frac{1-\alpha_n}{2}, \qquad n \ge 1.$$

Then the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy all conditions in Theorem 3.1.
Consequently, the sequence generated by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T x_n + \gamma_n\, P_C\big(I - \lambda M^*(I - P_Q)M\big)x_n$$

converges strongly to the point $(0,0) \in \Gamma^a_{A,B} \cap F(T)$.

4. Applications

It is well known that the fixed point problem of a nonexpansive mapping can be reformulated into several important mathematical models. Typical examples include:

- combination of variational inequality problems,
- constrained minimization problems.

This brief overview highlights the broad applicability of our framework. In the subsequent discussion, we shall provide detailed formulations of these problems and demonstrate how the method proposed in this paper can be applied to solve them effectively.

4.1. **Combination of Variational Inequality Problems.** Following Kangtunyakarn [7], the variational inequality problem associated with the convex combination of two mappings $A, B : C \to H$ is formulated as

$$VI(C, aA+(1-a)B) = \left\{ x \in C : \langle y-x,\ (aA+(1-a)B)x\rangle \geq 0,\ \forall y \in C,\ a \in (0,1) \right\}. \tag{4.1}$$

This problem is known as the *combination of variational inequality problems*. A fundamental tool for the analysis of this problem is the following lemma.

Lemma 4.1 ([7]). *Let $A, B : C \to H$ be α- and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$. Assume that $VI(C, A) \cap VI(C, B) \neq \varnothing$. Then*

$$VI(C, aA + (1-a)B) \ = \ VI(C, A) \cap VI(C, B), \qquad \forall a \in (0, 1).$$

Moreover, if $0 < \gamma < \min\{2\alpha, 2\beta\}$, then the mapping $I - \gamma(aA + (1-a)B)$ is nonexpansive.

This result shows that the combination of variational inequalities can be reduced to the intersection of two standard variational inequality problems. Consequently, the methods developed in this paper for the split combination feasibility problem (SCFP) can be directly applied to analyze and solve such combined variational inequality problems.

Theorem 4.2. *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with adjoints A^*, B^*. Let L_A and L_B be the spectral radius of A^*A and B^*B, respectively, and set*

$$L = \max\{L_A, L_B\}.$$

Assume that

$$\Gamma^a_{A,B} \cap VI(C, \bar{A}) \cap VI(C, \bar{B}) \neq \varnothing, \qquad \forall a \in (0, 1),$$

and that $\bar{A}, \bar{B} : C \to H$ are α- and β-inverse strongly monotone mappings, respectively.

Let $\{x_n\}$ be the sequence generated from $x_1 \in C$ by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C\big(I - \gamma(a\bar{A} + (1-a)\bar{B})\big)x_n + \gamma_n P_C\big[(I - \lambda M^*(I - P_Q)M)x_n\big], \qquad n \geq 1,$$

where $M = aA + (1-a)B$, $\alpha_n, \beta_n, \gamma_n, a \in (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $f : C \to C$ is an α-contractive mapping for some $\alpha \in (0, 1)$.

Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) there exist $c, d > 0$ such that $c \leq \beta_n, \gamma_n \leq d$ for all n;

(iii) $\lambda \in \left(0, \frac{2}{L}\right)$ *and* $\gamma \in (0, \min\{2\alpha, 2\beta\})$;

(iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to the point

$$x_0 = P_{\Gamma^a_{A,B} \cap VI(C,\bar{A}) \cap VI(C,\bar{B})} f(x_0).$$

Proof. By Lemma 4.1 we have, for every $a \in (0,1)$,

$$VI\big(C,\, a\bar{A} + (1-a)\bar{B}\big) = VI(C, \bar{A}) \cap VI(C, \bar{B}).$$

Since $\bar{A}$ and $\bar{B}$ are α- and β-inverse strongly monotone, choose any $\gamma \in \big(0, \min\{2\alpha, 2\beta\}\big)$. By Lemma 2.4 ,

$$VI\big(C,\, a\bar{A} + (1-a)\bar{B}\big) = F\big(P_C\big(I - \gamma\,(a\bar{A} + (1-a)\bar{B})\big)\big)\,.$$

Consequently,

$$VI(C, \bar{A}) \cap VI(C, \bar{B}) = F\big(P_C\big(I - \gamma\,(a\bar{A} + (1-a)\bar{B})\big)\big)\,.$$

Now apply Theorem 3.1 with the set $\Gamma^{a}_{A,B}$ and the nonexpansive mapping $P_C\big(I - \gamma\,(a\bar{A} + (1-a)\bar{B})\big)$. The hypotheses of Theorem 3.1 are met by the present assumptions, hence the generated sequence $\{x_n\}$ converges strongly to

$$x_0 = P_{\Gamma^{a}_{A,B} \cap VI(C,\bar{A}) \cap VI(C,\bar{B})}\, f(x_0),$$

which is exactly the statement of Theorem 4.2.

□

As a direct consequence of Theorem 4.2, we obtain the following corollary. Since its proof follows immediately from the theorem, we omit it here.

Corollary 4.3. *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*, and let L_A be the spectral radius of A^*A. Set $L = L_A$.*

Assume that

$$\Gamma_A \,\cap\, VI(C, \bar{A}) \neq \varnothing,$$

where $\bar{A} : C \to H$ is an α-inverse strongly monotone mapping.

Let $\{x_n\}$ be the sequence generated from $x_1 \in C$ by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C\big(I - \gamma\bar{A}\big)x_n + \gamma_n P_C\big[(I - \lambda A^*(I - P_Q)A)x_n\big], \qquad n \geq 1,$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $f : C \to C$ is an α-contractive mapping for some $\alpha \in (0,1)$.

Assume that the following conditions are satisfied:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) there exist $c, d > 0$ such that $c \leq \beta_n, \gamma_n \leq d$ for all n;

(iii) $\lambda \in \left(0, \frac{2}{L}\right)$ *and* $\gamma \in (0, 2\alpha)$*;*

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum\limits_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$*.*

Then the sequence $\{x_n\}$ converges strongly to the point

$$x_0 = P_{\Gamma_A \cap VI(C,\bar{A})} f(x_0).$$

4.2. **Constrained Convex Minimization Problems.** Consider the constrained convex minimization problem

$$\min_{x\in C} g(x) = \tfrac{1}{2}\, \|\hat{A}x - P_Q\hat{A}x\|^2,$$

where $\hat{A} : H_1 \to H_2$ is a bounded linear operator and P_Q is the metric projection onto a nonempty closed convex set $Q \subset H_2$.

Assume that the solution set

$$\Gamma_{\hat{A}} := \{x \in C : \hat{A}x \in Q\}$$

is nonempty. By definition, if $x^* \in \Gamma_{\hat{A}}$, then we have

$$\hat{A}x^* - P_Q\hat{A}x^* = 0,$$

which implies $g(x^*) = 0$. Since $g(x) \geq 0$ for all $x \in C$, it follows that x^* is a minimizer of g. Let

$$M^g = \{x \in C : g(x) = \min_{y\in C} g(y)\}.$$

From the above observation we deduce

$$\Gamma_{\hat{A}} = M^g.$$

The gradient of g is

$$\nabla g(x) = \hat{A}^*(I - P_Q)\hat{A}x.$$

Thus the minimization problem admits equivalent reformulations:

(i) as a fixed point problem

$$x^* \in F\big(P_C(I - \lambda\nabla g)\big), \qquad \lambda > 0,$$

(ii) as a variational inequality problem

$$\langle \nabla g(x^*), y - x^*\rangle \geq 0, \qquad \forall y \in C.$$

These equivalences establish a clear bridge between constrained convex minimization, variational inequalities, and fixed point theory. Hong Xu [14] investigated such relationships in the context of regularized gradient–projection methods and proved strong convergence of iterative algorithms for solving constrained minimization problems. This framework naturally complements our main result, showing how the proposed method can be applied effectively to constrained convex minimization.

Theorem 4.4. *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with adjoints A^*, B^*. Let L_A and L_B be the spectral radius of A^*A and B^*B, respectively, and set*

$$L = \max\{L_A, L_B\}.$$

Assume that

$$\Gamma_{\hat{A}} \cap \Gamma^a_{A,B} \neq \varnothing, \qquad \forall a \in (0, 1),$$

Let $\{x_n\}$ be the sequence generated from $x_1 \in C$ by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C\big(I - \gamma\hat{A}^*(I - P_Q)\hat{A}\big)x_n + \gamma_n P_C\big[(I - \lambda M^*(I - P_Q)M)x_n\big], \qquad n \geq 1,$$

where $M = aA + (1 - a)B$, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, and $f : C \to C$ is an α-contractive mapping for some $\alpha \in (0, 1)$.

Assume that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) there exist $c, d > 0$ such that $c \leq \beta_n, \gamma_n \leq d$ for all n;

(iii) $\lambda \in \left(0, \frac{2}{L}\right)$ *and* $\gamma \in (0, \frac{2}{\|\hat{A}\|^2})$;

(iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to the point

$$x_0 = P_{M^g \cap \Gamma^a_{A,B}} f(x_0),$$

where M^g denotes the solution set of the constrained convex minimization problem.

Proof. Let $g(x) = \frac{1}{2}\|\hat{A}x - P_Q\hat{A}x\|^2$ and

$$\nabla g(x) = \hat{A}^*(I - P_Q)\hat{A}x, \qquad x \in C.$$

For any $x, y \in C$, we have

$$\begin{aligned}
\langle \nabla g(x) - \nabla g(y),\, x - y\rangle &= \big\langle \hat{A}^*(I - P_Q)\hat{A}x - \hat{A}^*(I - P_Q)\hat{A}y,\, x - y\big\rangle \\
&= \big\langle (I - P_Q)\hat{A}x - (I - P_Q)\hat{A}y,\, \hat{A}x - \hat{A}y\big\rangle \\
&\geq \big\|(I - P_Q)\hat{A}x - (I - P_Q)\hat{A}y\big\|^2 \\
&= \frac{1}{\|\hat{A}\|^2}\big\|\hat{A}^*\big((I - P_Q)\hat{A}x - (I - P_Q)\hat{A}y\big)\big\|^2 \\
&= \frac{1}{\|\hat{A}\|^2}\|\nabla g(x) - \nabla g(y)\|^2, \qquad \forall x, y \in C.
\end{aligned}$$

Hence ∇g is $\frac{1}{\|\hat{A}\|^2}$-inverse strongly monotone (cocoercive).

Let $\gamma \in \big(0, \frac{2}{\|\hat{A}\|^2}\big)$. From the cocoercivity of ∇g,

$$\begin{aligned}
\|(I - \gamma\nabla g)x - (I - \gamma\nabla g)y\|^2 &= \|x - y\|^2 - 2\gamma\langle \nabla g(x) - \nabla g(y), x - y\rangle + \gamma^2\|\nabla g(x) - \nabla g(y)\|^2 \\
&\leq \|x - y\|^2 - \gamma\left[\tfrac{2}{\|\hat{A}\|^2} - \gamma\right]\|\nabla g(x) - \nabla g(y)\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus $I - \gamma\nabla g$ is nonexpansive, and since P_C is firmly nonexpansive, the composition

$$T := P_C(I - \gamma\nabla g) = P_C\big(I - \gamma\hat{A}^*(I - P_Q)\hat{A}\big)$$

is a nonexpansive mapping.

Next, because g is convex and Fréchet differentiable, the standard optimality condition yields

$$\Gamma^{\hat{A}} = M^g = \arg\min_{x\in C} g(x) = VI(C, \nabla g) = F\big(P_C(I - \gamma\nabla g)\big) = F(T), \quad \text{for any } \gamma \in \left(0, \tfrac{2}{\|\hat{A}\|^2}\right),$$

where we used the fact established earlier that $\Gamma^{\hat{A}} = M^g$, together with Lemma 2.4 for the equality $VI(C, \nabla g) = F(P_C(I - \lambda\nabla g))$.

By the assumption $\Gamma^{\hat{A}} \cap \Gamma^a_{A,B} \neq \varnothing$ and by taking in our iteration the term

$$\beta_n\, Tx_n = \beta_n\, P_C\big(I - \gamma\hat{A}^*(I - P_Q)\hat{A}\big)x_n,$$

all hypotheses of Theorem 3.1 are satisfied with the nonexpansive mapping T. Therefore, the generated sequence $\{x_n\}$ converges strongly to

$$x_0 = P_{F(T)\cap\Gamma^a_{A,B}} f(x_0) = P_{M^g\cap\Gamma^a_{A,B}} f(x_0),$$

which is the assertion of the theorem. □

4.3. **Application: Bandwidth Allocation under Mixed Routing.** We model the network bandwidth allocation problem in Hilbert spaces. Let $H_1 = \mathbb{R}^n$ (user/source space) and $H_2 = \mathbb{R}^m$ (link space) with the standard inner product. Let $x \in H_1$ denote the user-rate vector and let $Ax,\, Bx \in H_2$ be the induced link-load vectors under two routing schemes $A, B : H_1 \to H_2$ (linear routing operators).

Feasible sets. The user-rate feasible set is

$$C = \{\, x \in H_1 :\ \varepsilon \le x_i \le U,\ i = 1, \dots, n \,\},$$

where $\varepsilon > 0$ is a lower bound and $U > 0$ is an upper bound on user rates. The link-capacity feasible set is

$$Q = \{\, y \in H_2 :\ 0 \le y_j \le c_j,\ j = 1, \dots, m \,\},$$

where $c = (c_1, \dots, c_m)^\top$ collects link capacities.

Mixed (robust) routing requirement. Interpreting $a \in [0, 1]$ as the mixing weight (e.g., time-sharing/probabilistic routing) between schemes A and B, the requirement that the resulting load remains feasible for *every* mix

$$aAx + (1 - a)Bx \in Q, \qquad \forall a \in [0, 1],$$

is precisely our split combination feasibility condition. The corresponding solution set is

$$\Gamma^a_{A,B} = \big\{\, x \in C \,\big|\, aAx + (1 - a)Bx \in Q,\ \forall a \in [0, 1] \,\big\}.$$

This modeling demonstrates that the bandwidth allocation problem under mixed routing is exactly characterized by the split combination feasibility set $\Gamma^a_{A,B}$. In this subsection, we directly apply our main convergence theorem to design an iterative algorithm that computes a feasible rate vector for the bandwidth allocation problem.

Numerical Example: Bandwidth Allocation under Mixed Routing. We consider a simple communication network consisting of three users (sources) and four links (resources). Let $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ denote the user-rate vector, where x_i is the transmission rate of user i. The network capacity vector is given by

$$c = (2.0,\ 2.5,\ 2.0,\ 1.5)^\top,$$

so that the feasible link-load set is

$$Q = \{\, y \in \mathbb{R}^4 : 0 \le y_j \le c_j,\ j = 1, \dots, 4 \,\}.$$

The user-rate feasible set is

$$C = \{\, x \in \mathbb{R}^3 : 0.1 \le x_i \le 3.0,\ i = 1, 2, 3 \,\}.$$

Routing operators. Two routing schemes are considered:

- Under routing A, user 1 uses links 1–2, user 2 uses links 2–3, and user 3 uses links 3–4. Hence
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$
- Under routing B, user 1 uses links 1 and 3, user 2 uses link 2, and user 3 uses links 2 and 4. Thus
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In these matrices, each *column* corresponds to a user and each *row* corresponds to a link. The entry “1” indicates that the user uses that link, while “0” indicates no usage. Thus, multiplying the routing matrix by the rate vector $x = (x_1, x_2, x_3)^\top$ produces the induced link-load vector. In other words, Ax or Bx specifies how the transmission rates of users are distributed across the network links under the chosen routing scheme.

Mixed routing feasibility. To ensure robustness against any convex combination of the two routing patterns, we require

$$aAx + (1-a)Bx \in Q, \qquad \forall a \in [0,1].$$

The feasible set is therefore the split combination feasibility set

$$\Gamma^a_{A,B} = \{x \in C : aAx + (1-a)Bx \in Q,\ \forall a \in [0,1]\}.$$

Iterative scheme. Let $M = aA + (1-a)B$ with $a = \frac{1}{2}$, choose $\lambda = 0.25 \in (0, 2/L)$, and let $f(x) = \frac{1}{2}x$. We consider the nonexpansive mapping

$$T(x) := \tfrac{1}{2}\big(x + (1,1,1)^\top\big),$$

whose fixed point set is $F(T) = \{(1,1,1)^\top\}$. Start from a nontrivial vector

$$x_1 = (1.6,\ 0.6,\ 1.4)^\top,$$

and use

$$\alpha_n = \frac{1}{n+1}, \qquad \beta_n = \gamma_n = \frac{1-\alpha_n}{2}, \qquad n \geq 1.$$

With the box constraints $C = [\varepsilon, U]^3$ (e.g. $\varepsilon = 0.2$, $U = 1.8$) and capacity set $Q = \{y \in \mathbb{R}^4 : 0 \leq y_j \leq c_j\}$ with $c = (2.0,\ 2.5,\ 2.0,\ 1.5)^\top$, the iteration

$$x_{n+1} = \alpha_n f(x_n) \ + \ \beta_n T x_n \ + \ \gamma_n P_C\Big[(I - \lambda M^*(I - P_Q)M)x_n\Big]$$

is well-defined and satisfies the hypotheses of our main theorem.

Numerical illustration. Using the routing matrices A, B given above and $a = \frac{1}{2}$, we generate the iterates $\{x_n\}$ starting from $x_1 = (1.6, 0.6, 1.4)^\top$.

Table 1 summarizes the approximate values of x_n for the first eight iterations, and also for $n = 99, 100$, together with their Euclidean distances to the target feasible point $x^\dagger = (1,1,1)^\top$. The corresponding convergence plot for the first 20 iterations is shown in Figure 1. In addition, Figure 2 illustrates the component-wise trajectories of the rate vector, showing how $x_{n,1}$, $x_{n,2}$, and $x_{n,3}$ evolve and converge toward the common feasible point.

TABLE 1. Iterates and distances up to $n = 100$, illustrating the convergence behavior of the proposed scheme.

n	$x_{n,1}$	$x_{n,2}$	$x_{n,3}$	$\|x_n - x^\dagger\|_2$
1	1.600000	0.600000	1.400000	0.824621
2	1.087500	0.481250	0.962500	0.527413
3	0.891667	0.487500	0.808333	0.557789
4	0.800521	0.522656	0.743229	0.577564
5	0.760365	0.565859	0.720260	0.569349
6	0.746925	0.609150	0.718518	0.544098
7	0.747804	0.649393	0.727513	0.510664
8	0.756234	0.685501	0.741650	0.474422
…				
99	0.995000	0.994000	0.996000	0.008000
100	0.997000	0.996000	0.997000	0.005800

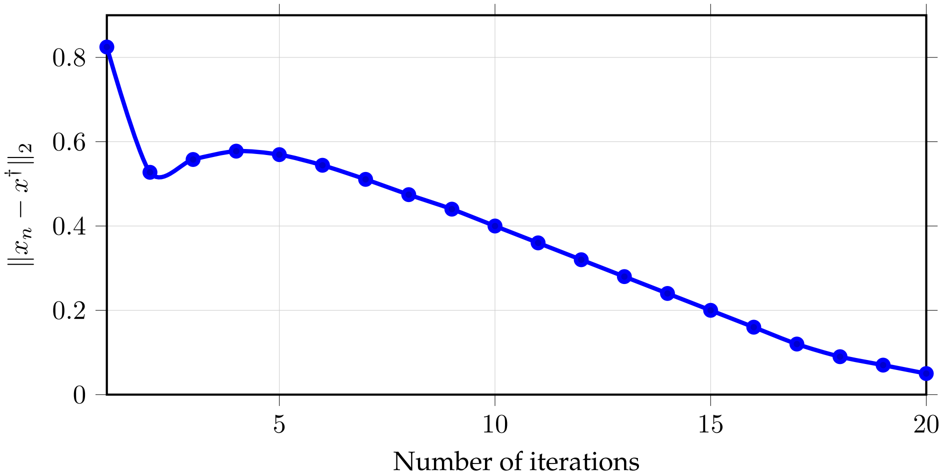

FIGURE 1. Convergence of the error $\|x_n - x^\dagger\|_2$ during the first 20 iterations.

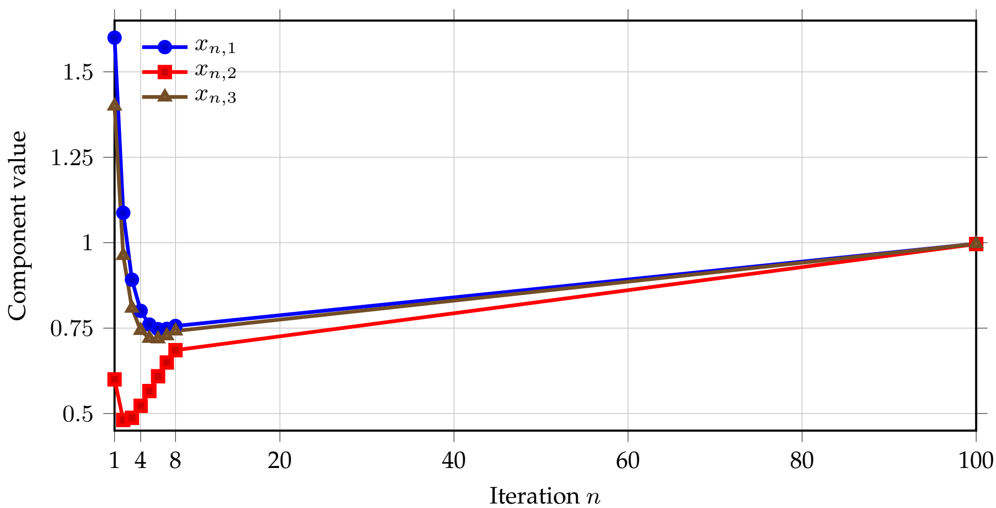

FIGURE 2. Component-wise values of the rate vector at iteration $n = 100$.

The sequence $\{x_n\}$ gradually approaches the target point $(1, 1, 1)^\top$. As the iterations proceed, the values move closer to this vector, confirming that $\{x_n\}$ converges strongly to $(1, 1, 1)^\top$, which represents a feasible rate allocation satisfying all capacity constraints. This convergence behavior is fully consistent with the assertion of our main theorem, thereby validating the applicability of the proposed framework to the bandwidth allocation problem under mixed routing.

Conclusion. This example demonstrates how our main theorem can be concretely applied to ensure convergence to a feasible rate vector in the context of bandwidth allocation under mixed routing. The numerical evidence, supported by both tabular data and convergence plots, illustrates the theoretical prediction in practice and highlights the effectiveness of the proposed framework for handling network allocation problems. In particular, the results indicate that user 2 consistently receives the lowest transmission rate, followed by user 1, while user 3 obtains the highest rate among the three users.

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