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A Strongly Convergent Self-Adaptive Double Inertial Forward-Backward Method for Inclusion and Fixed Point Problems in Hilbert spaces

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ABSTRACT. This work introduces a novel forward-backward splitting algorithm to find a common solution of a monotone inclusion problem and a fixed point problem for a nonexpansive mapping in Hilbert spaces. The algorithm's architecture synthesizes a double inertial extrapolation step with a Mann viscosity-type framework and a self-adaptive inertial parameterization. A primary contribution of this structure is the establishment of a strong convergence theorem, thereby addressing a gap where previous double inertial methods typically guaranteed only weak convergence. The generality of the main theorem is demonstrated by deriving strongly convergent corollaries for variational inequality, convex minimization, and split feasibility problems. Numerical experiments validate the algorithm's performance and, crucially, demonstrate that the self-adaptive mechanism achieves near-optimal performance without the need for manual hyperparameter tuning.

1. INTRODUCTION

Let H be a real Hilbert space and the mapping $B : H \to 2^H$ be a set-valued operator. The variational inclusion problem, which was introduced by Martinet [18], is the problem of finding a point $x^* \in H$ such that

$$0 \in Bx^*. \tag{1.1}$$

when B is a maximal monotone operator, the elements in the solution set of problem (1.1) are called the zeros of a maximal monotone operator. The variational inclusion problems have been used as mathematical programming models to study optimization problems arising in economics, finance, transportation, network, and engineering sciences. The method for solving problem (1.1) is the following proximal point method: for any $x_1 \in H$,

$$x_{n+1} = J^B_{\lambda_n} x_n, \qquad \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ is the resolvent of the maximal monotone operator B corresponding to λ_n; see [13, 17, 37, 41] for more information.

A general type of problem (1.1) is the problem of finding a point $x^* \in H$ such that

$$0 \in Ax^* + Bx^*, \tag{1.3}$$

where $A : H \to H$ is a single-valued operator and $B : H \to 2^H$ is a set-valued operator. When A and B are monotone operators, the elements in the solution set of problem (1.3) are called the zeros of the sum of monotone operators; see [4, 9, 19, 24, 35] and the references therein. Note that, if A is a continuous and monotone operator and B is a maximal monotone operator, then $A + B$ is a maximal monotone operator; see [26]. A popular

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iterative method for solving problem (1.3) is the so-called the forward-backward splitting method, which defines a sequence $\{x_n\}$ by the following algorithm: for a given $x_1 \in H$,

$$x_{n+1} = J^B_{\lambda_n}(I - \lambda_n A)x_n, \qquad \forall n \in \mathbb{N}, \tag{1.4}$$

where $A : H \to H$ and $B : H \to 2^H$ are maximal monotone operators, and $\{\lambda_n\}$ is a sequence of positive real numbers; see Passty [21].

In addition, the inertial technique was first presented by Polyak in 1964, to speed up the rate of convergence; see [23]. This technique is a two-step iterative method, in which each iteration involves the previous two iterates. Consequently, many authors considered the inertial method because of this faster convergence rate property of the algorithm; see [2, 10, 11, 22, 25, 34] for more information.

In 2001, Alvarez and Attouch [1] proposed the inertial proximal point algorithm for solving problem (1.1): for arbitrary $x_0, x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= J^B_{\lambda_n} y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where $\{\lambda_n\}$ is non-decreasing sequence and $\{\mu_n\} \subset [0, 1)$ satisfy with $\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\|^2 < \infty$, and presented the weakly convergence results.

In 2015, Lorenz and Pock [15] studied the monotone inclusion problem (1.3) and suggested the inertial forward-backward algorithm as follows: for arbitrary $x_0, x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= J^B_{\lambda_n}(I - \lambda_n A)y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.6}$$

where $\{\lambda_n\}$ is a positive real sequence and $\{\mu_n\} \subset [0, 1)$. By suitable conditions, they proved that $\{x_n\}$ converges weakly to a solution of problem (1.3).

In 2018, Cholamjiak et al. [10] proposed the Halpern type inertial forward-backward method for finding the solution of problem (1.3): for arbitrary $u, x_0, x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} &= \alpha_n u + \beta_n y_n + \gamma_n J^B_{\lambda_n}(I - \lambda_n A)y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.7}$$

where $\{\mu_n\} \subset [0, \mu)$ with $\mu \in [0, 1)$, the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$ and some suitable conditions. They proved that $\{x_n\}$ converges strongly to a solution of problem (1.3).

In 2020, Tan et al. [34] proposed the modified Mann viscosity method with the inertial algorithm for solving fixed point problems. Let S be nonexpansive mapping, for arbitrary $x_0, x_1 \in H$,

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= \varphi_n z_n + (1 - \varphi_n)Sz_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\varphi_n\}$ in $(0, 1)$, and $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. The convergence result, they showed that the sequence $\{x_n\}$ converges strongly to $z = P_{F(S)}f(z)$. These contributions are part of a broader, active effort to combine inertial steps with strong-convergence frameworks (see, e.g., [28])

Furthermore, more researchers are interested in studying and applying the double inertial method to algorithms to improve their efficiency and produce better results; see [12, 27, 29, 36] for more details. Dong et al. [12] proposed the general Mann iteration

method with double inertial as follows: let S be a nonexpansive mapping, for arbitrary $x_0, x_1 \in H$,

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= (1-\varphi_n)z_n + \varphi S y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.9}$$

where $\{\mu_n\} \subset [0, \mu]$ and $\{\theta_n\} \subset [0, \theta]$ are nondecreasing with $\mu_1 = \theta_1 = 0$, and $\mu, \theta \in [0, 1)$. By suitable conditions, the sequence $\{x_n\}$ generated by (1.9) converges weakly to a point of $F(S)$.

In 2023, Suantai et al. [29] suggested a forward-backward-forward algorithm with double inertial for solving monotone inclusions as follows: for arbitrary $x_0, x_1 \in H$,

$$\begin{aligned} u_n &= x_n + \mu_n(x_n - x_{n-1}), \\ z_n &= u_n + \theta_n(u_n - x_{n-1}), \\ y_n &= J^B_{\lambda_n}(I - \lambda_n A)x_n, \\ x_{n+1} &= y_n - \lambda_n(Ay_n - Az_n), \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.10}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\phi\|z_n - y_n\|}{\|Az_n - Ay_n\|}, \lambda_n + \gamma_n\right\}, & \text{if } Az_n - Ay_n \neq 0; \\ \lambda_n + \gamma_n, & \text{otherwise,} \end{cases}$$

when $\lambda_1 > 0$, $\phi \in (0, 1)$, $\{\gamma_n\}$ is a nonnegative sequence such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $\{\mu_n\}$ and $\{\theta_n\}$ are real positive sequences. By suitable conditions, they proved that the sequence $\{x_n\}$ converges weakly to a solution of problem (1.3).

It is well known that the concept of double inertia is not merely an extension but is at the forefront of a significant and active area of current research. There is a recent surge of interest in studying and applying double, dual, or quasi-inertial methods to enhance algorithms for solving complex problems. This is evidenced by numerous recent publications applying such techniques to variational inequalities [6, 7, 8, 40] and fixed point problems [39], with applications in fields from image processing to medical classification. This manuscript contributes directly to this highly contemporary line of inquiry.

We note that the preceding review highlights two distinct research trajectories: (1) The pursuit of accelerated convergence rates via multi-step inertial techniques, as seen in the double inertial methods of Dong et al. [12] and Suantai et al. [29], which notably only established *weak* convergence. (2) The separate pursuit of *strong* convergence, which is critical for many applications, typically achieved by integrating a viscosity or Halpern-type framework, as demonstrated by Cholamjiak et al. [10] and Tan et al. [34]. A clear gap exists in unifying these two lines of inquiry. The problem of whether a *double* inertial forward-backward algorithm can be structured within a *viscosity-type* framework to guarantee *strong* convergence for the important problems remains an open and compelling question. This manuscript is dedicated to resolving this.

In this paper, motivated and inspired by the above literature, we will consider the monotone inclusion problem and fixed point problem, that is, the problem of finding a point $x^* \in H$ such that

$$x^* \in F(S) \cap (A + B)^{-1}0 =: \Omega, \tag{1.11}$$

where $S : H \to H$ is a nonexpansive mapping. We aim to suggest the algorithms for finding a common solution of problem (1.11) and provide some suitable conditions to guarantee that the constructed sequence $\{x_n\}$ converges strongly to a point in Ω.

2. PRELIMINARIES

In this section, we present relevant notations and definitions, as well as lemmas that can be used to prove our main theorem. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $\{x_n\}$ be a sequence in H, such that the strong and weak convergence of the sequence $\{x_n\}$ to x are given by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $S : H \to H$ be a mapping. Then S is said to be

(i) *Lipschitz* if there exists $L \geq 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \qquad \forall x, y \in H.$$

The number L is called a Lipschitz constant. Also, if $L \in [0, 1)$ we say S is a contraction, if $L = 1$ we say S is a nonexpansive.

(ii) *Firmly nonexpansive* if

$$\|Sx - Sy\|^2 \leq \langle x - y, Sx - Sy \rangle, \qquad \forall x, y \in H.$$

(iii) *σ-inverse strongly monotone (σ-ism)* if for a positive real number σ,

$$\langle Sx - Sy, x - y \rangle \geq \sigma\|Sx - Sy\|^2, \qquad \forall x, y \in H.$$

Below we have included some important properties for our proof.

Lemma 2.1. [4, 38] *The following are true:*
(i) If $A : H \to H$ is σ-ism, then A is $\frac{1}{\sigma}$-Lipschitz continous and monotone mapping.
(ii) If $A : H \to H$ is σ-ism and $\lambda \in (0, \sigma]$, then $S := I - \lambda A$ is firmly nonexpansive.

Let $B : H \to 2^H$ be a set-valued mapping. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. Recall that B is said to be monotone if

$$\langle x - y, u - v \rangle \geq 0, \qquad \forall x, y \in D(B), u \in Bx, v \in By.$$

A monotone mapping B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. To a maximal monotone operator $B : H \to 2^H$ and $\lambda > 0$, its resolvent J_λ^B is defined by

$$J_\lambda^B := (I + \lambda B)^{-1} : H \to D(B).$$

Notice that the resolvent J_λ^B is a single-valued and firmly nonexpansive mapping, and $F(J_\lambda^B) = B^{-1}0 \equiv \{x \in H : 0 \in Bx\}, \forall \lambda > 0$; see [30, 33].

Lemma 2.2. [3] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and $A : C \to H$ be an operator. If $B : H \to 2^H$ is a maximal monotone operator, then $F\big(J_\lambda^B(I - \lambda A)\big) = (A + B)^{-1}0$.*

The following fundamental results are needed in our proof.

For each $x, y, z \in H$, the following facts are valid for inner product spaces,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2, \tag{2.12}$$

and

$$\begin{aligned} \|\alpha x + \gamma y + \delta z\|^2 &= \alpha\|x\|^2 + \gamma\|y\|^2 + \delta\|z\|^2 \\ &\quad -\alpha\gamma\|x - y\|^2 - \alpha\delta\|x - z\|^2 - \gamma\delta\|y - z\|^2, \end{aligned} \tag{2.13}$$

for any $\alpha, \gamma, \delta \in [0, 1]$ such that $\alpha + \gamma + \delta = 1$; see [20, 30].

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called a metric projection of H onto C; see [32]. The following property of P_C is well known and useful:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C.$$

We also use the following lemmas to prove the main theorems.

Lemma 2.3. [31] *Let C be a closed convex subset of a Hilbert space H and $S : C \to C$ be a nonexpansive mapping. Then, $U := I - S$ is demiclosed, that is, $x_n \rightharpoonup x_0$ and $Ux_n \to y_0$ imply $Ux_0 = y_0$.*

Lemma 2.4. [14, 37] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n + c_n, \qquad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying
(i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} b_n \leq 0$;
(iii) $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < \infty$.
Then, $a_n \to 0$ as $n \to \infty$.

3. Main Results

This section will present the following assumptions and the double inertial forward-backward method with a Mann viscosity-type algorithm to solve the monotone inclusion problem and fixed point of nonexpansive mapping, problem (1.11).

(A1) $A : H \to H$ is a σ-inverse strongly monotone operator;
(A2) $B : H \to 2^H$ is a maximal monotone operator;
(A3) $S : H \to H$ is a nonexpansive mapping;
(A4) $f : H \to H$ is a contraction mapping with coefficient $\omega \in (0, 1)$.

Algorithm 3.1. *Let $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\varphi_n\}$ be sequences in $(0, 1)$ with $\alpha_n + \gamma_n + \delta_n = 1$ and the initial $x_0 = u_0, x_1 \in H$ be arbitraries, define*

$$\begin{aligned} u_n &= x_n + \mu_n(x_n - x_{n-1}), \\ z_n &= u_n + \theta_n(u_n - u_{n-1}), \\ y_n &= \varphi_n x_n + (1 - \varphi_n)J^B_{\lambda_n}(I - \lambda_n A)z_n, \\ x_{n+1} &= \alpha_n f(x_n) + \gamma_n x_n + \delta_n S y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{3.14}$$

where $\lambda_n \in (0, \sigma)$, and $\{\mu_n\}$ and $\{\theta_n\}$ are defined as follow: $0 \leq \mu_n \leq \bar{\mu}_n$ such that

$$\bar{\mu}_n = \begin{cases} \min\left\{\mu(\|x_n - x_{n-1}\|), \dfrac{\beta_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ 0, & \text{otherwise}, \end{cases} \tag{3.15}$$

and $0 \leq \theta_n \leq \bar{\theta}_n$ such that

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta(\|u_n - u_{n-1}\|), \dfrac{\beta_n}{\|u_n - u_{n-1}\|}\right\}, & \text{if } u_n \neq u_{n-1}; \\ 0, & \text{otherwise}, \end{cases} \tag{3.16}$$

when $\{\beta_n\}$ is a positive sequence satisfying $\beta_n = o(\alpha_n)$ and $\mu, \theta : [0, \infty) \to [0, 1)$.

Remark 3.1. *From Algorithm 3.1, we see that* $\mu_n \leq \frac{\beta_n}{\|x_n - x_{n-1}\|}$ *and* $\theta_n \leq \frac{\beta_n}{\|u_n - u_{n-1}\|}$*, respectively. These imply that* $\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\beta_n}{\alpha_n}$ *and* $\frac{\theta_n}{\alpha_n}\|u_n - u_{n-1}\| \leq \frac{\beta_n}{\alpha_n}$*;* $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$*, we get*

$$\lim_{n\to\infty} \frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| = 0 \tag{3.17}$$

and also have

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|u_n - u_{n-1}\| = 0. \tag{3.18}$$

Remark 3.2. *A significant methodological advantage of Algorithm 3.1 lies in the definition of the inertial parameters in (3.15) and (3.16). The upper bounds* $\overline{\mu}_n$ *and* $\overline{\theta}_n$ *are not fixed constants but are determined by functions* $\mu(\cdot)$ *and* $\theta(\cdot)$ *that depend on the norm of the previous step. This construction provides a self-adaptive inertial mechanism. It allows the algorithm to dampen the inertial effect if the iterates begin to oscillate or make a large jump (i.e., when* $\|x_n - x_{n-1}\|$ *is large), thereby enhancing stability. This contrasts with many algorithms in the literature that employ a fixed inertial parameter, which can be seen as a special case of this more general and adaptive formulation.*

Remark 3.3. *The construction of the iterate* y_n *in (3.14) is a key feature of the algorithm. It employs a Mann-type average between the current iterate* x_n *and the forward-backward step evaluated at the double-inertial point* z_n*. This choice can be interpreted as a stabilizing mechanism, balancing the acceleration provided by the* μ_n *and* θ_n *terms with an anchor to the non-extrapolated sequence* $\{x_n\}$*. The subsequent proof in Theorem 3.1 validates that this specific construction is sufficient to achieve strong convergence under the stated conditions.*

Using the above assumptions and applying Algorithm 3.1 to our main result, we obtain the strong convergence theorem (Theorem 3.1) as follows.

Theorem 3.1. *Let* H *be a real Hilbert space and let* $\{x_n\}$ *be generated by Algorithm 3.1. Suppose that the assumptions (A1)-(A4) hold,* $\Omega \neq \emptyset$ *and the following control conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) There exist a positive real number a *with* $0 < a \leq \gamma_n$ *and* $0 < a \leq \delta_n$*, for each* $n \in \mathbb{N}$*;*

(C3) There exist positive real numbers b_1 *and* b_2 *with* $0 < b_1 \leq \varphi_n \leq b_2 < 1$*, for each* $n \in \mathbb{N}$*.*

Then, $\{x_n\}$ *converges strongly to* $p \in \Omega$*, where* $p = P_\Omega f(p)$*.*

Proof. Firstly, we prove that $\{x_n\}$ is a bounded sequence. Let $z \in \Omega$. Then, $z \in F(S)$, implies $Sz = z$. Moreover, we have $z \in (A + B)^{-1}0$, this implies that $J^B_{\lambda_n}(I - \lambda_n A)z = z$. Now, consider the following relation,

$$\begin{aligned} \|z_n - z\| &= \|u_n + \theta_n(u_n - u_{n-1}) - z\| \\ &\leq \|u_n - z\| + \theta_n\|u_n - u_{n-1}\|, \end{aligned} \tag{3.19}$$

for each $n \in \mathbb{N}$. We will get the first term of (3.19) as follows,

$$\begin{aligned} \|u_n - z\| &= \|x_n + \mu_n(x_n - x_{n-1}) - z\| \\ &\leq \|x_n - z\| + \mu_n\|x_n - x_{n-1}\|, \end{aligned} \tag{3.20}$$

for each $n \in \mathbb{N}$. Substitute (3.20) in (3.19) and using (3.17) and (3.18), we obtain

$$\begin{aligned} \|z_n - z\| &\leq \|x_n - z\| + \mu_n\|x_n - x_{n-1}\| + \theta_n\|u_n - u_{n-1}\| \\ &\leq \|x_n - z\| + \alpha_n\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| + \alpha_n\frac{\theta_n}{\alpha_n}\|u_n - u_{n-1}\| \\ &\leq \|x_n - z\| + \alpha_n K_1, \end{aligned} \tag{3.21}$$

for some $K_1 > 0$. By the definition of y_n and (3.21), we have

$$\begin{aligned}
\|y_n - z\| &= \big\|\varphi_n x_n + (1-\varphi_n)J^B_{\lambda_n}(I-\lambda_n A)z_n - z\big\| \\
&\leq \varphi_n\|x_n - z\| + (1-\varphi_n)\big\|J^B_{\lambda_n}(I-\lambda_n A)z_n - z\big\| \\
&\leq \varphi_n\|x_n - z\| + (1-\varphi_n)\|z_n - z\| \\
&\leq \varphi_n\|x_n - z\| + (1-\varphi_n)\|x_n - z\| + (1-\varphi_n)\alpha_n K_1 \\
&\leq \|x_n - z\| + (1-\varphi_n)\alpha_n K_1, \qquad (3.22)
\end{aligned}$$

for each $n \in \mathbb{N}$. Then, using the definition of x_{n+1} and (3.22), we obtain

$$\begin{aligned}
\|x_{n+1} - z\| &= \big\|\alpha_n f(x_n) + \gamma_n x_n + \delta_n S y_n - z\big\| \\
&\leq \alpha_n\big\|f(x_n) - z\big\| + \gamma_n\|x_n - z\| + \delta_n\|Sy_n - z\| \\
&\leq \alpha_n\big\|f(x_n) - f(z)\big\| + \alpha_n\big\|f(z) - z\big\| + \gamma_n\|x_n - z\| + \delta_n\|y_n - z\| \\
&\leq \alpha_n\omega\|x_n - z\| + \alpha_n\big\|f(z) - z\big\| + \gamma_n\|x_n - z\| \\
&\quad +\delta_n\|x_n - z\| + \delta_n(1-\varphi_n)\alpha_n K_1 \\
&= \big(1 - \alpha_n(1-\omega)\big)\|x_n - z\| + \alpha_n(1-\omega)\left(\frac{\big\|f(z) - z\big\| + K_1}{1-\omega}\right) \\
&\leq \max\left\{\|x_n - z\|, \frac{\big\|f(z) - z\big\| + K_1}{1-\omega}\right\} \\
&\vdots \\
&\leq \max\left\{\|x_1 - z\|, \frac{\big\|f(z) - z\big\| + K_1}{1-\omega}\right\}, \qquad (3.23)
\end{aligned}$$

for each $n \in \mathbb{N}$. Therefore, we conclude that $\big\{\|x_n - z\|\big\}$ is a bounded sequence, implying $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, and $\{f(x_n)\}$ are also bounded.

Next, we note that $P_\Omega f(\cdot)$ is a contraction mapping. Let p be a unique fixed point of $P_\Omega f(\cdot)$, i.e. $p = P_\Omega f(p)$. In the following, we will show the necessary relationships to prove the convergence theorem. Consider, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|u_n - p\|^2 &= \|x_n + \mu_n(x_n - x_{n-1}) - p\|^2 \\
&\leq \|x_n - p\|^2 + \mu_n^2\|x_n - x_{n-1}\|^2 + 2\mu_n\|x_n - p\|\|x_n - x_{n-1}\| \\
&\leq \|x_n - p\|^2 + \mu_n\|x_n - x_{n-1}\|\big(\mu_n\|x_n - x_{n-1}\| + 2\|x_n - p\|\big) \\
&\leq \|x_n - p\|^2 + K_2\mu_n\|x_n - x_{n-1}\|, \qquad (3.24)
\end{aligned}$$

where $K_2 = 3\sup_n\big\{\mu\|x_n - x_{n-1}\|, \|x_n - p\|\big\} > 0$.

$$\begin{aligned}
\|z_n - p\|^2 &= \|u_n + \theta_n(u_n - u_{n-1}) - p\|^2 \\
&\leq \|u_n - p\|^2 + \theta_n^2\|u_n - u_{n-1}\|^2 + 2\theta_n\|u_n - p\|\|u_n - u_{n-1}\| \\
&\leq \|u_n - p\|^2 + \theta_n\|u_n - u_{n-1}\|\big(\theta_n\|u_n - u_{n-1}\| + 2\|u_n - p\|\big) \\
&\leq \|u_n - p\|^2 + K_3\theta_n\|u_n - u_{n-1}\|, \qquad (3.25)
\end{aligned}$$

where $K_3 = 3\sup_n\big\{\theta\|u_n - u_{n-1}\|, \|u_n - p\|\big\} > 0$, for each $n \in \mathbb{N}$. By substituting (3.24) into the inequality (3.25), we get

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 + K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\|, \qquad (3.26)$$

for each $n \in \mathbb{N}$. By the definition of y_n and (3.26), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|\varphi_n x_n + (1-\varphi_n)J^B_{\lambda_n}(I-\lambda_n A)z_n - p\right\|^2 \\
&\leq \varphi_n\|x_n - p\|^2 + (1-\varphi_n)\left\|J^B_{\lambda_n}(I-\lambda_n A)z_n - p\right\|^2 \\
&\quad -\varphi_n(1-\varphi_n)\left\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\right\|^2 \\
&\leq \varphi_n\|x_n - p\|^2 + (1-\varphi_n)\|z_n - p\|^2 \\
&\leq \varphi_n\|x_n - p\|^2 + (1-\varphi_n)\|x_n - p\|^2 \\
&\quad +K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\| \\
&\leq \|x_n - p\|^2 + K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\|, \qquad (3.27)
\end{aligned}$$

for each $n \in \mathbb{N}$. In addition, by substituting the obtained relation (3.27) into the following, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\langle \alpha_n f(x_n) + \gamma_n x_n + \theta_n S y_n - p, x_{n+1} - p\big\rangle \\
&= \alpha_n\big\langle f(x_n) - f(p), x_{n+1} - p\big\rangle + \alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\quad +\gamma_n\big\langle x_n - p, x_{n+1} - p\big\rangle + \delta_n\big\langle Sy_n - p, x_{n+1} - p\big\rangle \\
&\leq \frac{\alpha_n}{2}\Big(\|f(x_n) - f(p)\|^2 + \|x_{n+1} - p\|^2\Big) \\
&\quad +\frac{\gamma_n}{2}\Big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\Big) \\
&\quad +\frac{\delta_n}{2}\Big(\|Sy_n - p\|^2 + \|x_{n+1} - p\|^2\Big) \\
&\quad +\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\leq \left(\frac{\alpha_n\omega^2}{2} + \frac{\gamma_n}{2}\right)\|x_n - p\|^2 + \frac{\delta_n}{2}\|y_n - p\|^2 + \frac{1}{2}\|x_{n+1} - p\|^2 \\
&\quad +\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&\leq \left(\frac{\alpha_n\omega^2 + \gamma_n}{2}\right)\|x_n - p\|^2 + \frac{\delta_n}{2}\|x_n - p\|^2 \\
&\quad +\frac{\delta_n}{2}\Big\{K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\|\Big\} \\
&\quad +\frac{1}{2}\|x_{n+1} - p\|^2 + \alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle,
\end{aligned}$$

for each $n \in \mathbb{N}$. This gives,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \big(1 - \alpha_n(1-\omega^2)\big)\|x_n - p\|^2 + K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\| \\
&\quad +2\alpha_n\big\langle f(p) - p, x_{n+1} - p\big\rangle \\
&= \big(1 - \alpha_n(1-\omega^2)\big)\|x_n - p\|^2 + \alpha_n(1-\omega^2)T_n, \qquad (3.28)
\end{aligned}$$

where $T_n = \dfrac{K_2}{1-\omega^2}\dfrac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| + \dfrac{K_3}{1-\omega^2}\dfrac{\theta_n}{\alpha_n}\|u_n - u_{n-1}\| + \dfrac{2}{1-\omega^2}\big\langle f(p) - p, x_{n+1} - p\big\rangle$.

Next, we prove the desired result by considering the following two cases.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - p\|\}$ is monotonically non-increasing for all $n \in \mathbb{N}$. Since a sequence $\{\|x_n - p\|\}$ is bounded, it is a convergent sequence. By using (3.26) and the definition of y_n, we also get

$$\begin{aligned}
\|y_n - p\|^2 &= \big\|\varphi_n x_n + (1-\varphi_n)J^B_{\lambda_n}(I-\lambda_n A)z_n - p\big\|^2 \\
&\leq \varphi_n\|x_n - p\|^2 + (1-\varphi_n)\big\|J^B_{\lambda_n}(I-\lambda_n A)z_n - p\big\|^2 \\
&\quad -\varphi_n(1-\varphi_n)\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\|^2 \\
&\leq \varphi_n\|x_n - p\|^2 + (1-\varphi_n)\|z_n - p\|^2 - \varphi_n(1-\varphi_n)\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\|^2 \\
&\leq \varphi_n\|x_n - p\|^2 + (1-\varphi_n)\|x_n - p\|^2 + K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\| \\
&\quad -\varphi_n(1-\varphi_n)\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\|^2 \\
&\leq \|x_n - p\|^2 + K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\| \\
&\quad -\varphi_n(1-\varphi_n)\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\|^2, \qquad (3.29)
\end{aligned}$$

for each $n \in \mathbb{N}$. Furthermore, we use (2.13) and (3.29), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|\alpha_n f(x_n) + \gamma_n x_n + \delta_n S y_n - p\big\|^2 \\
&= \big\|\alpha_n\big(f(x_n) - p\big) + \gamma_n(x_n - p) + \delta_n(Sy_n - p)\big\|^2 \\
&= \alpha_n\|f(x_n) - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|Sy_n - p\|^2 \\
&\quad -\alpha_n\gamma_n\|f(x_n) - x_n\|^2 - \alpha_n\delta_n\|f(x_n) - Sy_n\|^2 - \gamma_n\delta_n\|x_n - Sy_n\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|x_n - p\|^2 \\
&\quad +K_2\mu_n\|x_n - x_{n-1}\| + K_3\theta_n\|u_n - u_{n-1}\| \\
&\quad -\varphi_n(1-\varphi_n)\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\|^2,
\end{aligned}$$

for each $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\varphi_n(1-\varphi_n)\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +\alpha_n K_2\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| + \alpha_n K_3\frac{\theta_n}{\alpha_n}\|u_n - u_{n-1}\|. \qquad (3.30)
\end{aligned}$$

By using Remark 3.1 and conditions (C1) and (C3), we obtain

$$\lim_{n\to\infty}\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\| = 0. \qquad (3.31)$$

Now, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_n - y_n\| &= \big\|x_n - \varphi_n x_n - (1-\varphi_n)J^B_{\lambda_n}(I-\lambda_n A)z_n\big\| \\
&\leq (1-\varphi_n)\big\|x_n - J^B_{\lambda_n}(I-\lambda_n A)z_n\big\|.
\end{aligned}$$

Using (3.31), it follows that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \qquad (3.32)$$

Next, consider

$$\begin{aligned}
\|z_n - x_n\| &= \|u_n + \theta_n(u_n - u_{n-1}) - x_n\| \\
&= \|x_n + \mu_n(x_n - x_{n-1}) + \theta_n(u_n - u_{n-1}) - x_n\| \\
&\leq \mu_n\|x_n - x_{n-1}\| + \theta_n\|u_n - u_{n-1}\| \\
&= \alpha_n\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| + \alpha_n\frac{\theta_n}{\alpha_n}\|u_n - u_{n-1}\|,
\end{aligned}$$

for each $n \in \mathbb{N}$. By (3.17), (3.18) and condition (C1), it follows that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.33}$$

From the relation

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \alpha_n\|f(x_n)-p\|^2+\gamma_n\|x_n-p\|^2+\delta_n\|Sy_n-p\|^2\\ &\quad -\alpha_n\gamma_n\|f(x_n)-x_n\|^2-\alpha_n\delta_n\|f(x_n)-Sy_n\|^2-\gamma_n\delta_n\|x_n-Sy_n\|^2,\end{aligned}$$

we have

$$\begin{aligned}\gamma_n\delta_n\|x_n-Sy_n\|^2 &\le \alpha_n\|f(x_n)-p\|^2+\gamma_n\|x_n-p\|^2+\delta_n\|y_n-p\|^2\\ &\quad -\|x_{n+1}-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\gamma_n\|x_n-p\|^2+\delta_n\|x_n-p\|^2\\ &\quad +K_2\mu_n\|x_n-x_{n-1}\|+K_3\theta_n\|u_n-u_{n-1}\|-\|x_{n+1}-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad \alpha_nK_2\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|+\alpha_nK_3\frac{\theta_n}{\alpha_n}\|u_n-u_{n-1}\|,\end{aligned} \tag{3.34}$$

for each $n \in \mathbb{N}$. By using conditions (C1) and (C2), we obtain

$$\lim_{n\to\infty} \|x_n - Sy_n\| = 0. \tag{3.35}$$

Moreover, we will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Consider

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_nf(x_n)+\gamma_nx_n+\delta_nSy_n-x_n\|\\ &\le \alpha_n\|f(x_n)-x_n\|+\delta_n\|Sy_n-x_n\|,\end{aligned}$$

for each $n \in \mathbb{N}$. By using (3.35) and condition (C1), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.36}$$

Since the sequence $\{x_n\}$ is bounded on H, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to $x^* \in H$. The next step is to prove that $x^* \in \Omega$.

To show $x^* \in (A+B)^{-1}0$, we first consider

$$\begin{aligned}\left\|x_n - J^B_{\lambda_n}(I-\lambda_nA)x_n\right\| &\le \left\|x_n - J^B_{\lambda_n}(I-\lambda_nA)z_n\right\| + \left\|J^B_{\lambda_n}(I-\lambda_nA)z_n - J^B_{\lambda_n}(I-\lambda_nA)x_n\right\|\\ &\le \left\|x_n - J^B_{\lambda_n}(I-\lambda_nA)z_n\right\| + \|z_n - x_n\|,\end{aligned}$$

for each $n \in \mathbb{N}$. By using (3.31) and (3.33), it follows that

$$\lim_{n\to\infty} \left\|x_n - J^B_{\lambda_n}(I-\lambda_nA)x_n\right\| = 0. \tag{3.37}$$

Next, consider the following relation

$$\begin{aligned}\left\|x^* - J^B_{\lambda_n}(I-\lambda_nA)x^*\right\|^2 &\le \left\langle x^* - J^B_{\lambda_n}(I-\lambda_nA)x^*, x^* - x_{n_j}\right\rangle\\ &\quad +\left\langle x^* - J^B_{\lambda_n}(I-\lambda_nA)x^*, x_{n_j} - J^B_{\lambda_n}(I-\lambda_nA)x_{n_j}\right\rangle\\ &\quad +\left\langle x^* - J^B_{\lambda_n}(I-\lambda_nA)x^*, J^B_{\lambda_n}(I-\lambda_nA)x_{n_j} - J^B_{\lambda_n}(I-\lambda_nA)x^*\right\rangle,\end{aligned} \tag{3.38}$$

for each $j \in \mathbb{N}$. By $x_{n_j} \rightharpoonup x^*$ and (3.37), we obtain

$$\lim_{j\to\infty} \left\|x^* - J^B_{\lambda_n}(I-\lambda_nA)x^*\right\| = 0.$$

Thus, we get $x^* = J^B_{\lambda_n}(I-\lambda_nA)x^*$, implies that $x^* \in (A+B)^{-1}0$.

To show that $x^* \in F(S)$, we consider the following relation

$$\|y_n - Sy_n\| \le \|y_n - x_n\| + \|x_n - Sy_n\|,$$

for each $n \in \mathbb{N}$. By using (3.32) and (3.35), we have

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \tag{3.39}$$

Using (3.39) and together with $y_{n_j} \rightharpoonup x^*$, for each $j \in \mathbb{N}$, we obtain from Lemma 2.3 that $x^* \in F(S)$. From the above results, it shows that $x^* \in \Omega$.

Finally, we will prove that the sequence $\{x_n\}$ converges strongly to $p = P_\Omega f(p)$. Now, the sequence $\{x_n\}$ is bounded, and we have $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. Without loss of generality, we may assume that a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ converges weakly to $x^* \in H$. It follows that

$$\begin{aligned} \limsup_{n\to\infty} \frac{2}{1-\omega^2}\langle f(p) - p, x_{n+1} - p\rangle &= \lim_{j\to\infty} \frac{2}{1-\omega^2}\langle f(p) - p, x_{n_j+1} - p\rangle \\ &= \frac{2}{1-\omega^2}\langle f(p) - p, x^* - p\rangle \le 0. \end{aligned} \tag{3.40}$$

By using (3.40) and together with (3.17) and (3.18), we obtain

$$\limsup_{n\to\infty} \left(\frac{K_2}{1-\omega^2}\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| + \frac{K_3}{1-\omega^2}\frac{\theta_n}{\alpha_n}\|u_n - u_{n-1}\| + \frac{2}{1-\omega^2}\langle f(p) - p, x_{n+1} - p\rangle \right) \le 0.$$

That is from (3.28), we get $\limsup_{n\to\infty} T_n \le 0$. By using Lemma 2.4 to (3.28), we can conclude that $\|x_n - p\| \to 0$, as $n \to \infty$. Therefore $x_n \to p$, as $n \to \infty$.

Case 2: Suppose that $\{\|x_n - p\|\}$ is not a monotonically decreasing sequence for all $n \in \mathbb{N}$. Now, we set $\Gamma_n = \|x_n - p\|$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \quad \Gamma_k \le \Gamma_{k+1}\},$$

for all $n \in \mathbb{N}$. Then, we obtain $\{\tau(n)\}$ is a nondecreasing sequence, with $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \quad \forall n \in \mathbb{N};$$

see [16]. From this result, we get $\|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \le 0$, for each $n \in \mathbb{N}$.

Consider, from (3.30), it follows that

$$\begin{aligned} \varphi_{\tau(n)}(1-\varphi_{\tau(n)})\|x_{\tau(n)} - J^B_{\lambda_{\tau(n)}}(I - \lambda_{\tau(n)}A)z_{\tau(n)}\|^2 &\le \alpha_{\tau(n)}\|f(x_{\tau(n)}) - p\|^2 \\ &\quad + \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \\ &\quad + \alpha_{\tau(n)}K_2\frac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\| \\ &\quad + \alpha_{\tau(n)}K_3\frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\|u_{\tau(n)} - u_{\tau(n)-1}\| \\ &\le \alpha_{\tau(n)}\|f(x_{\tau(n)}) - p\|^2 \\ &\quad + \alpha_{\tau(n)}K_2\frac{\mu_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\| \\ &\quad + \alpha_{\tau(n)}K_3\frac{\theta_{\tau(n)}}{\alpha_{\tau(n)}}\|u_{\tau(n)} - u_{\tau(n)-1}\|, \end{aligned} \tag{3.41}$$

for each $n \in \mathbb{N}$.

Similar as in **Case 1**, we also have

$$\lim_{n\to\infty} \|x_{\tau(n)} - J^B_{\lambda_{\tau(n)}}(I - \lambda_{\tau(n)}A)z_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$$

and

$$\limsup_{n\to\infty} T_{\tau(n)} \leq 0.$$

By the sequence $\{x_{\tau(n)}\}$ is bounded, we can find a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to $x^* \in F(S)\cap(A+B)^{-1}0$. From the relation in (3.28), we obtain

$$\|x_{\tau(n)+1} - p\|^2 \quad \leq \quad \big(1 - \alpha_{\tau(n)}(1-\omega^2)\big)\|x_{\tau(n)} - p\|^2 + \alpha_{\tau(n)}(1-\omega^2)T_{\tau(n)},$$

for each $n \in \mathbb{N}$. Consequently, we have

$$\begin{aligned} \alpha_{\tau(n)}(1-\omega^2)\|x_{\tau(n)} - p\|^2 &\leq \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \\ &\quad +\alpha_{\tau(n)}(1-\omega^2)T_{\tau(n)}, \\ &\leq \alpha_{\tau(n)}(1-\omega^2)T_{\tau(n)}. \end{aligned} \tag{3.42}$$

Since $\alpha_{\tau(n)}(1-\omega^2) > 0$, it follows from (3.42) that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - p\|^2 \leq 0.$$

This implies

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\|^2 = 0,$$

and also

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\| = 0. \tag{3.43}$$

Moreover, we use $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$ and (3.43), we get

$$\|x_{\tau(n)+1} - p\| \leq \|x_{\tau(n)+1} - x_{\tau(n)}\| + \|x_{\tau(n)} - p\| \to 0, \tag{3.44}$$

as $n \to \infty$. In addition, if $\tau(n) < n$, we also have $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, because $\Gamma_{\tau(n)+1} \geq \Gamma_j$ for $\tau(n)+1 \leq j \leq n$. Consequently, we obtain

$$0 \leq \Gamma_n \leq \max\left\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\right\} = \Gamma_{\tau(n)+1},$$

for each $n \in \mathbb{N}$. By using (3.44), we have $\lim_{n\to\infty} \Gamma_n = 0$. Therefore, we can conclude that $\{x_n\}$ converges strongly to p. This completes the proof. □

3.1. **Applications of Theorem 3.1.** In this subsection, we demonstrate the utility of our main theorem by applying it to several well-known optimization problems.

3.1.1. *Variational Inequality Problem.* Consider the normal cone to C at $u \in C$ defined as $N_C(u) = \{z \in H : \langle z, y-u\rangle \leq 0, \forall y \in C\}$. We note that N_C is a maximal monotone operator (A2). By setting $B := N_C$, problem (1.3) is reduced to the variational inequality problem (VIP) of finding $x^* \in C$ such that $\langle Ax^*, x - x^*\rangle \geq 0, \forall x \in C$. We denote the solution set by $VIP(C,A)$. In this case, the resolvent $J^B_{\lambda_n}$ becomes the metric projection P_C. The common solution problem (1.11) is now to find $x^* \in F(S) \cap VIP(C,A) =: \Omega_1$.

Applying Theorem 3.1 directly yields the following corollary:

Corollary 3.1. *Let A be σ-ism (A1) and S, f satisfy (A3) and (A4). Let $\{x_n\}$ be generated by the algorithm:*

$$\begin{cases} u_n &= x_n + \mu_n(x_n - x_{n-1}) \\ z_n &= u_n + \theta_n(u_n - u_{n-1}) \\ y_n &= \varphi_n x_n + (1-\varphi_n)P_C(I - \lambda_n A)z_n \\ x_{n+1} &= \alpha_n f(x_n) + \gamma_n x_n + \delta_n S y_n \end{cases}, \quad \forall n \in \mathbb{N}.$$

If $\Omega_1 \neq \emptyset$ and conditions (C1)-(C3) hold, then $\{x_n\}$ converges strongly to $p \in \Omega_1$, where $p = P_{\Omega_1} f(p)$.

3.1.2. *Convex Minimization Problem.* Let $h : H \to \mathbb{R}$ be a convex and Fréchet differentiable function, and let $A := \nabla h$. If ∇h is v-Lipschitz, then A is $(1/v)$-ism, satisfying (A1). By setting $B := N_C$ as in the VIP case, problem (1.3) becomes the convex minimization problem of finding $x^* \in C$ such that $x^* \in \arg\min_{x \in C} h(x)$. The common solution problem (1.11) is to find $x^* \in F(S) \cap \arg\min_{x \in C} h(x) =: \Omega_2$.

Corollary 3.2. *Let S, f satisfy (A3) and (A4), and let $A = \nabla h$ be $(1/v)$-ism. Let $\{x_n\}$ be generated by the algorithm:*

$$\begin{cases} u_n &= x_n + \mu_n(x_n - x_{n-1}) \\ z_n &= u_n + \theta_n(u_n - u_{n-1}) \\ y_n &= \varphi_n x_n + (1-\varphi_n)P_C(I - \lambda_n \nabla h)z_n \\ x_{n+1} &= \alpha_n f(x_n) + \gamma_n x_n + \delta_n S y_n \end{cases}, \quad \forall n \in \mathbb{N}.$$

If $\Omega_2 \neq \emptyset$, $\lambda_n \in (0, 1/v)$, and conditions (C1)-(C3) hold, then $\{x_n\}$ converges strongly to $p \in \Omega_2$, where $p = P_{\Omega_2} f(p)$.

3.1.3. *Split Feasibility Problem.* Let H_1, H_2 be two real Hilbert spaces, $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex sets, and $L : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) [5] is to find $x^* \in C$ such that $Lx^* \in Q$. Let $A = L^*(I - P_Q)L$, where L^* is the adjoint of L. It is known [33] that A is $(1/\|L\|^2)$-ism, satisfying (A1). Let $B = N_C$ (A2). Then $(A+B)^{-1}0$ is the solution set of the SFP. The common solution problem (1.11) is to find $x^* \in F(S) \cap C \cap L^{-1}Q =: \Omega_3$.

Corollary 3.3. *Let S, f satisfy (A3) and (A4). Let $\{x_n\}$ be generated by the algorithm:*

$$\begin{cases} u_n &= x_n + \mu_n(x_n - x_{n-1}) \\ z_n &= u_n + \theta_n(u_n - u_{n-1}) \\ y_n &= \varphi_n x_n + (1-\varphi_n)P_C(I - \lambda_n L^*(I - P_Q)L)z_n \\ x_{n+1} &= \alpha_n f(x_n) + \gamma_n x_n + \delta_n S y_n \end{cases}, \quad \forall n \in \mathbb{N}.$$

If $\Omega_3 \neq \emptyset$, $\lambda_n \in (0, 1/\|L\|^2)$, and conditions (C1)-(C3) hold, then $\{x_n\}$ converges strongly to $p \in \Omega_3$, where $p = P_{\Omega_3} f(p)$.

4. Numerical Experiments

In this section, we will consider some numerical experiments to illustrate the use of Theorem 3.1.

Example 4.1. *Let $H = \mathbb{R}^2$ be equipped with the Euclidean norm. For each $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H$, we consider the following two norms:*

$$\|x\|_1 = |x_1| + |x_2| \quad \textit{and} \quad \|x\|_\infty = \max\{|x_1|, |x_2|\}.$$

Let a function $g : H \to \mathbb{R}$, which is defined by

$$g(x) = \|x\|_1, \qquad \forall x \in H.$$

Now, we have a subdifferential operator of g is

$$\partial g(x) = \big\{z \in H : \langle x, z\rangle = \|x\|_1, \|z\|_\infty \leq 1\big\}, \qquad \forall x \in H.$$

Since g is a convex function, then $\partial g(\cdot)$ is a maximal monotone operator. Moreover, for each $\lambda > 0$, we have

$$J_\lambda^{\partial g}(x) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H : u_i = x_i - \big(\min\{|x_i|, \lambda\}\big) sgn(x_i), \ \ for \ \ i = 1, 2 \right\},$$

where $sgn(\cdot)$ stands for the signum function.

Next, let $\tilde{x} := \begin{pmatrix} 7 \\ 4 \end{pmatrix}$ and $\bar{x} := \begin{pmatrix} 11 \\ -8 \end{pmatrix}$ be two fixed vectors in H. We consider the 1-Lipschitz operator P_Q, where

$$Q := \big\{u \in H : \langle \tilde{x}, u\rangle \leq -11\big\},$$

and the operator P_C, where C is the following nonempty convex subsets of H:

$$C := \big\{u \in H : \langle \bar{x}, u\rangle \leq 22\big\}.$$

Now, we get $F(P_C) = C$. In addition, we consider a contraction mapping $f := \begin{bmatrix} \frac{1}{20} & 0 \\ 0 & \frac{1}{30} \end{bmatrix}$. Under the above settings, we will consider the problem to find a point

$$x^* \in C \cap (P_Q + \partial g)^{-1} 0. \tag{4.45}$$

Note that, the solution set of problem (4.45) is $\Omega := \left\{ \begin{pmatrix} x \\ \frac{4x-3}{7} \end{pmatrix} \in H : \frac{3}{4} \leq x \leq \frac{130}{45} \right\}$. We determine the results using the stopping criterion by $\dfrac{\|x_{n+1} - x_n\|}{\max\{1, \|x_n\|\}} \leq 1.0e^{-06}$.

We set the control parameters as $\beta_n = \dfrac{1}{(2n+100)^2}$, $\alpha_n = \dfrac{1}{2n+100}$, $\gamma_n = \dfrac{1-\alpha_n}{2}$ and $\delta_n = \dfrac{1-\alpha_n}{2}$.

We notice that the maximal monotone operator as $B := \partial g$, satisfying condition (A2). We set the single-valued operator $A := P_Q$. Because P_Q is a metric projection onto a closed convex set, it is firmly nonexpansive, which implies it is 1-inverse strongly monotone (1-ism), thereby satisfying condition (A1). Next, we define the nonexpansive mapping $S := P_C$, noting that its fixed point set is $F(P_C) = C$, which satisfies condition (A3). The operator f is a given contraction mapping, satisfying (A4). Consequently, finding a point $x^* \in C \cap (P_Q + \partial g)^{-1} 0$ is mathematically equivalent to finding $x^* \in F(S) \cap (A + B)^{-1} 0$. Furthermore, the selected control parameters $\{\alpha_n\}, \{\gamma_n\}, \{\delta_n\}$, and $\{\beta_n\}$ are constructed to satisfy conditions (C1)-(C3). Thus, all hypotheses of Theorem 3.1 are fulfilled, guaranteeing that the sequence $\{x_n\}$ generated by Algorithm 3.1 will converge strongly to the solution.

Experiment 1: Comparison of Inertial Structures (Table 1)

We first investigate the performance of different inertial structures. We test the four cases defined as:

Case 1: $\mu_n = \bar{\mu}_n$, $\theta_n = \bar{\theta}_n$ (Double Inertial);
Case 2: $\mu_n = \bar{\mu}_n$, $\theta_n \to 0$ (Double Inertial with diminishing upper bound);
Case 3: $\mu_n = \bar{\mu}_n$, $\theta_n = 0$ (Single Inertial);
Case 4: $\mu_n = 0$, $\theta_n = 0$ (Non-Inertial).

TABLE 1. Numerical experiments for different step size parameters μ_n and θ_n of Algorithm 3.1 with some initial points.

Case →	*Case 1*		*Case 2*		*Case 3*		*Case 4*	
# x_0, x_1 ↓	*Iters*	*Sol*	*Iters*	*Sol*	*Iters*	*Sol*	*Iters*	*Sol*
IP 1	*271*	$\begin{pmatrix}0.715407\\0\end{pmatrix}$	*278*	$\begin{pmatrix}0.716117\\0\end{pmatrix}$	*271*	$\begin{pmatrix}0.715440\\0\end{pmatrix}$	*3,371*	$\begin{pmatrix}0.746598\\0\end{pmatrix}$
IP 2	*271*	$\begin{pmatrix}0.715407\\0\end{pmatrix}$	*285*	$\begin{pmatrix}0.716811\\0\end{pmatrix}$	*271*	$\begin{pmatrix}0.715440\\0\end{pmatrix}$	*3,371*	$\begin{pmatrix}0.746598\\0\end{pmatrix}$
IP 3	*3,548*	$\begin{pmatrix}0.746825\\0\end{pmatrix}$	*3,555*	$\begin{pmatrix}0.746831\\0\end{pmatrix}$	*3,550*	$\begin{pmatrix}0.746827\\0\end{pmatrix}$	*3,552*	$\begin{pmatrix}0.746829\\0\end{pmatrix}$
IP 4	*4,907*	$\begin{pmatrix}0.747701\\0\end{pmatrix}$	*4,927*	$\begin{pmatrix}0.747705\\0\end{pmatrix}$	*4,909*	$\begin{pmatrix}0.747702\\0\end{pmatrix}$	*4,911*	$\begin{pmatrix}0.747703\\0\end{pmatrix}$

For Cases 1, 2 and 3, we use the self-adaptive functions $\mu(x) = \theta(x) = \dfrac{e^x - 1}{e^x}$, *as praised in Remark 3.2. The fixed parameters* $\lambda_n = 0.5$ *and* $\varphi_n = 0.5$ *are used. We consider four different initial points as follows.*

IP 1: $x_0 = (0,0)^\top, x_1 = (1,1)^\top$;
IP 2: $x_0 = (-1,-1)^\top, x_1 = (1,1)^\top$;
IP 3: $x_0 = (-1,10)^\top, x_1 = (1,10)^\top$;
IP 4: $x_0 = (-1,-1)^\top, x_1 = (10,10)^\top$.

From Table 1, we found that for any initial points, Case 1 (the double-adaptive method) gives a better convergence rate, especially when compared to Case 3 (single-adaptive) and Case 4 (non-inertial). This establishes the double-inertial structure as the most effective.

Experiment 2: Validating the Self-Adaptive Mechanism (Table 2 & 3)

Given that the Case 1 structure is superior, we now validate the practical benefit of the self-adaptive mechanism (Remark 3.2). A major challenge in optimization is the difficult task of hyper-parameter tuning.

The chosen adaptive function $\mu(x) = \theta(x) = \dfrac{e^x - 1}{e^x} = 1 - e^{-x}$, *where* x *is the norm of the previous step (e.g.,* $x = \|x_n - x_{n-1}\|$*), is designed to solve this problem. It has two distinct, critical behaviors:*

(1) ***Acceleration (Large Steps):*** *When the algorithm is far from the solution, the step size* x *is large. As* $x \to \infty$, *the function* $\mu(x) \to 1$. *This allows the algorithm to use a large inertial parameter to accelerate convergence.*
(2) ***Damping (Small Steps):*** *As the algorithm converges, the step size* $x \to 0$. *A Taylor expansion shows* $\mu(x) = 1 - e^{-x} \approx 1 - (1 - x) = x$. *The inertial parameter automatically and intelligently vanishes, damping the algorithm to prevent overshoot and stabilize convergence.*

We now present numerical evidence of this dual behavior. We compare the performance of our adaptive method (from Table 1, Case 1) against the same algorithm using five different fixed inertial parameters (i.e., $\mu(x) = c = \theta(x)$*). The results of these fixed-parameter tests are in Table 2.*

We compute the average of iterations for these fixed-parameter tests and summarize them in Table 3.

The results in Table 3 are highly insightful and perfectly confirm our analysis of the adaptive function. They show that finding a good fixed parameter is non-trivial:

TABLE 2. Influence of the parameter μ of Algorithm 3.1 with some initial point.

$\mu \rightarrow$	$\mu = 1.0e^{-10}$		$\mu = 1.0e^{-06}$		$\mu = 1.0e^{-03}$		$\mu = 0.5$		$\mu = 0.99$	
# $x_0, x_1 \downarrow$	*Iters*	*Sol*	*Iters*	*Sol*	*Iters*	*Sol*	*Iters*	*Sol*	*Iters*	*Sol*
IP 1	*271*	$\begin{pmatrix} 0.715440 \\ 0 \end{pmatrix}$	*271*	$\begin{pmatrix} 0.715440 \\ 0 \end{pmatrix}$	*271*	$\begin{pmatrix} 0.715429 \\ 0 \end{pmatrix}$	*3.371*	$\begin{pmatrix} 0.746598 \\ 0 \end{pmatrix}$	*3,371*	$\begin{pmatrix} 0.746598 \\ 0 \end{pmatrix}$
IP 2	*271*	$\begin{pmatrix} 0.715440 \\ 0 \end{pmatrix}$	*271*	$\begin{pmatrix} 0.715440 \\ 0 \end{pmatrix}$	*271*	$\begin{pmatrix} 0.715429 \\ 0 \end{pmatrix}$	*3.371*	$\begin{pmatrix} 0.746598 \\ 0 \end{pmatrix}$	*3,371*	$\begin{pmatrix} 0.746598 \\ 0 \end{pmatrix}$
IP 3	*3,550*	$\begin{pmatrix} 0.746827 \\ 0 \end{pmatrix}$	*3,550*	$\begin{pmatrix} 0.746827 \\ 0 \end{pmatrix}$	*3,548*	$\begin{pmatrix} 0.746825 \\ 0 \end{pmatrix}$	*3,548*	$\begin{pmatrix} 0.746825 \\ 0 \end{pmatrix}$	*3,548*	$\begin{pmatrix} 0.746825 \\ 0 \end{pmatrix}$
IP 4	*4,909*	$\begin{pmatrix} 0.747702 \\ 0 \end{pmatrix}$	*4,909*	$\begin{pmatrix} 0.747702 \\ 0 \end{pmatrix}$	*4,907*	$\begin{pmatrix} 0.747701 \\ 0 \end{pmatrix}$	*4,907*	$\begin{pmatrix} 0.747701 \\ 0 \end{pmatrix}$	*4,907*	$\begin{pmatrix} 0.747701 \\ 0 \end{pmatrix}$

TABLE 3. Average iterations for different fixed μ values (from Table 2).

Parameter Value ($\mu = c$)	***Average Iterations***
$\mu = 1.0e^{-10}$	*2250.25*
$\mu = 1.0e^{-06}$	*2250.25*
$\mu = 1.0e^{-03}$	*2249.25*
$\mu = 0.5$	*3799.25*
$\mu = 0.99$	*3799.25*

- *A poor choice of a large, fixed parameter ($\mu = 0.5$ or $\mu = 0.99$) leads to very slow convergence (3799.25 avg. iterations). This is the "Acceleration" behavior without the "Damping": the large, fixed parameter causes persistent oscillation and overshoot near the solution.*
- *An optimally-tuned small, fixed parameter ($\mu = 1.0e^{-03}$) is very fast (2249.25 avg. iterations). This is the "Damping" behavior, which appears to be critical for this problem's convergence.*
- *Crucially, the data from our adaptive function in Table 1 (Case 1)* ***automatically achieved this near-optimal performance*** *(2249.25 avg. iterations, from Table 1 data).*

This is the key advantage: the function's "Damping" property (Behavior 2) allowed it to intelligently and automatically mimic the optimally-tuned fixed parameter as it approached the solution. The proposed self-adaptive mechanism provides acceleration when needed and automatic damping, achieving the best performance without requiring the user to perform these difficult tuning experiments.

Experiment 3: Influence of φ_n (Table 4)

In Table 4, we focus on the parameter φ_n. We return to the superior Case 1 setup (using the adaptive functions $\mu(x)$ and $\theta(x)$) and test three different fixed values for φ_n.

From the presented results in Table 4, it can be seen that a smaller value of parameter φ_n leads to faster convergence.

5. CONCLUSIONS

This paper has introduced a new double inertial forward-backward algorithm (Algorithm 3.1) for the common problem of finding a zero of the sum of two monotone operators and a fixed point of a nonexpansive mapping (1.11). The algorithm's design integrates two key novelties: a double inertial extrapolation step and a self-adaptive inertial parameterization (Remark 3.2), combined within a Mann viscosity-type iteration.

TABLE 4. Influence of the parameter φ_n of Algorithm 3.1 with some initial point.

$\varphi_n \rightarrow$	$\varphi_n = 0.01$		$\varphi_n = 0.5$		$\varphi_n = 0.9$	
# $x_0, x_1 \downarrow$	*Iters*	*Sol*	*Iters*	*Sol*	*Iters*	*Sol*
IP 1	*240*	$\begin{pmatrix}0.730256\\0\end{pmatrix}$	*271*	$\begin{pmatrix}0.715407\\0\end{pmatrix}$	*418*	$\begin{pmatrix}0.643651\\0\end{pmatrix}$
IP 2	*240*	$\begin{pmatrix}0.730256\\0\end{pmatrix}$	*271*	$\begin{pmatrix}0.715407\\0\end{pmatrix}$	*418*	$\begin{pmatrix}0.643651\\0\end{pmatrix}$
IP 3	*2,719*	$\begin{pmatrix}0.747907\\0\end{pmatrix}$	*3,548*	$\begin{pmatrix}0.746825\\0\end{pmatrix}$	*5,544*	$\begin{pmatrix}0.739947\\0\end{pmatrix}$
IP 4	*3,413*	$\begin{pmatrix}0.748330\\0\end{pmatrix}$	*4,907*	$\begin{pmatrix}0.747701\\0\end{pmatrix}$	*12,317*	$\begin{pmatrix}0.745507\\0\end{pmatrix}$

The primary theoretical contribution is the proof of strong convergence (Theorem 3.1). This result formally addresses a noted gap in the literature, where similar multi-step inertial methods have, until now, generally only established weak convergence. We demonstrated the utility of this main theorem by establishing strongly convergent corollaries for the variational inequality, convex minimization, and split feasibility problems.

Furthermore, the numerical experiments in Section 4 provide a practical validation of the algorithm's design. The comparisons in Table 1 illustrate the computational advantage of the proposed double-inertial structure (Case 1) over single-inertial (Case 3) and non-inertial (Case 4) variants.

More importantly, the experiments validate the practical benefit of the self-adaptive mechanism. The results in Table 2 and Table 3 show that finding an optimal fixed parameter is a non-trivial tuning problem, with poor choices (e.g., $\mu = 0.5$) leading to significantly slower convergence. Our adaptive algorithm (Case 1, Table 1) automatically achieved this near-optimal performance without requiring any manual hyperparameter tuning. This demonstrates a key practical advantage, enhancing the algorithm's robustness and ease of use.

Future research may be directed toward analyzing the algorithm's convergence rate or relaxing the control conditions required for its implementation.

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REFERENCES

[1] Alvarez, F.; Attouch, H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Wellposedness in optimization and related topics (Gargnano, 1999). *Set-Valued Anal.* **9** (2001), 3—11.

[2] Anh, P.K.; Thong, D.V.; Dung, V.T. A strongly convergent Mann-type inertial algorithm for solving split variational inclusion problems. *Optim. Eng.* **2020** (2020).

[3] Aoyama, K.; Kimura, Y.; Takahashi, W.; Toyoda, M. On a strongly nonexpansive sequence in Hilbert spaces. *J. Nonlinear Convex Anal.* **8** (2007), 471–489.

[4] Boikanyo, O.A. The viscosity approximation forward-backward splitting method for zeros of the sum of monotone operators. *Abstr. Appl. Anal.* **2016** (2016).

[5] Censor, Y.; Elfving, T. A multiprojection algorithm using Bregman projections in product space. *Numer. Algorithms* **8** (1994), 221–239.

[6] Ceng, L.C.; Cholamjiak, P.; Inkrong, P.; Yao, J.C.; Zhao, X. Dual-Inertial Extragradient Rule for Pseudomonotonic Variational Inequalities with Split CFPP Constraint with Multiple Output Sets. *Commun. Nonlinear Sci. Numer. Simul.* (2025), 109212.
[7] Ceng, L.C.; Ghosh, D.; Rehman, H.; Yao, J.C.; Zhao, X. On quasi-inertial double extragradient-type iterations for variational inequalities and Bregman relatively asymptotically nonexpansive mapping. *Optim. Eng.* 26 (2) (2025), 1069-1111.
[8] Ceng, L.C.; Wen, C.F.; Yao, J.C. On Quasi-Inertial Double Extragradient-Type Iterations for Variational Inequalities with CFPP Constraint Involving Bregman Relatively Asymptotically Nonexpansive Mapping. *Numer. Funct. Anal. Optim.* 46 (6) (2025), 425-460.
[9] Cholamjiak, P.; Hieu, D.V.; Cho, Y.J. Relaxed forward-backward splitting methods for solving variational inclusions and applications. *J. Sci. Comput.* **88** (2021).
[10] Cholamjiak, W.; Cholamjiak, P.; Suantai, S. An inertial forward-backward splitting method for solving inclusion problems in Hilbert spaces. *J. Fixed Point Theory Appl.* **20** (2018).
[11] Dechboon, P.; Adamu, A.; Kumam, P. A generalized Halpern-type forward-backward splitting algorithm for solving variational inclusion problems. *AIMS Mathematics* **8** (2023), 11037–11056.
[12] Dong, Q.L.; Cho, Y.J.; Rassias, T.M. General inertial Mann algorithms and their convergence analysis for nonexpansive mappings. *Appl. Nonlinear Anal.* **134** (2018), 175–191.
[13] Eckstein, J.; Bertsckas, D.P. On the Douglas Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Appl. Math. Mech.-Engl. Math. Programming* **55** (1992), 293–318.
[14] Liu, L.S. Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. *J. Math. Anal. Appl.* **194** (1995), 114–125.
[15] Lorenz, D.; Pock, T. An inertial forward-backward algorithm for monotone inclusions. *J. Math. Imaging Vis.* **51** (2015), 311–325.
[16] Maingé, P.E. Strong convergence of projected subgradient methods for nonsmooth and non- strictly convex minimization. *Set-Valued Anal.* **16** (2008), 899–912.
[17] Marino, G.; Xu, H.K. Convergence of generalized proximal point algorithm *Comm. Pure Appl. Anal.* **3** (2004), 791–808.
[18] Martinet, B. Régularisation d'inéquations variationnelles par approximations successives. *Revue Francaise d'Informatique et de Recherche Opérationnelle* **3** (1970), 154–158.
[19] Moudafi, A.; Thera, M. Finding a zero of the sum of two maximal monotone operators. *J. Optimiz. Theory App.* **94** (1997), 425–448.
[20] Osilike, M.O.; Igbokwe, D.I. Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations. *Comput. Math. Appl.* **40** (2000), 559–567.
[21] Passty, G.B. Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. *J. Math. Anal. appl.* **72** (1979), 383–390.
[22] Petrot, N.; Khonchaliew, M.; Suwannaprapa, M. Modified inertial Tseng type method for zeros of the sum of monotone operators in Hilbert spaces. *CARPATHIAN J. MATH.* **40** (2024), 399–417.
[23] Polyak, B. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* **4** (1964), 1–17.
[24] Qin, X.; Cho, S.Y.; Wang, L. A regularization method for treating zero points of the sum of two monotone operators. *Fixed Point Theory A.* **2014** (2014).
[25] Rattanaseeha, K..; Khonchaliew, M. Modified inertial subgradient extragradient algorithm with self-adaptive step sizes for solving split equilibrium problems. J. Nonlinear Anal. Optim. 14(2) (2023), 71-85.
[26] Rockafellar, R.T. On the maximality of sums of nonlinear monotone operators. *T. Am. Math. Soc.* **149** (1970), 75-88.
[27] Saejia, P.; Suantai, S. A new two-step inertial algorithm for solving convex bilevel optimization problems with application in data classification problems. *AIMS Mathematics*, **9** (2024) 8476–8496.
[28] Shehu, Y.; Ogwo, G.T. "Strong convergence of inertial forward–backward splitting method for solving monotone inclusion problems." *Calcolo* 57, 23 (2020).
[29] Suantai, S.; Inkrong, P.; Cholamjiak, P. Forward-backward-forward algorithms involving two inertial terms for monotone inclusions. *Comput. Appl. Math.* **42** (2023).
[30] Takahashi, W. *Introduction to nonlinear and convex analysis*, Yokohama Publishers, Yokohama, 2009.
[31] Takahashi, W. *Nonlinear functional analysis: Fixed point theory and its applications*, Yokohama Publishers, Yokohama, 2000.
[32] Takahashi, W.; Toyoda, M. Weak convergence theorems for nonexpansive mappings and monotone mappings. *J. Optimiz. Theory App.* **118** (2003), 417–428.
[33] Takahashi, W.; Xu, H.K.; Yao, J.C. Iterative methods for generalized split feasibility problems in Hilbert spaces. *Set-Valued Var. Anal.* **23** (2015), 205–221.
[34] Tan, B.; Zhou, Z.; Li, S. Strong convergence of modified inertial Mann algorithms for nonexpansive mappings. *Mathematics* **8** (2020).

[35] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *Siam J. Control Optim.* **38** (2000), 431–446.
[36] Wang, Zb.; Lei, Zy.; Long, X.; Chen, Zy. A modified Tseng splitting method with double inertial steps for solving monotone inclusion problems. *J Sci Comput* **96** (2023).
[37] Xu, H.K. Iterative algorithms for nonlinear operators. *J. London Math. Soc.* **66** (2002), 240–256.
[38] Xu, H.K. Averaged mappings and the gradient-projection algorithm. *J. Optim. Theory Appl.* **150** (2011), 360–378.
[39] Yajai, W.; Kankam, K.; Yao, J.C.; Cholamjiak, W. A double inertial embedded modified S-iteration algorithm for nonexpansive mappings: A classification approach for lung cancer detection. *Commun. Nonlinear Sci. Numer. Simul.* (2025), 108978.
[40] Yao, Y.; Adamu, A.; Shehu, Y.; Yao, J.C. Forward-reflected-backward algorithms with double inertial extrapolations for variational inequalities with applications to image processing. *Numer. Algorithms* (2025), 1-36.
[41] Yao, Y.; Noor, M.A. On convergence criteria of generalized proximal point algorithms. *J. Comput. Appl. Math.* **217** (2008), 46–55.

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