CARPATHIAN J. MATH.
Volume **43** (2027), No. 1,
Pages 167-181

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2027.01.10

An Averaged Mann-Type Algorithm with Inertial Technique for Solving Fixed-Point Problems

SUPARAT BAIYA[1], SOMYOT PLUBTIENG[2] AND TADCHAI YUYING[3]

ABSTRACT. This paper introduces an averaged Mann-type algorithm incorporating an inertial technique and a self-adaptive step size to solve common fixed-point problems in Hilbert spaces. The proposed method involves nonexpansive and β-demicontractive mappings in Hilbert spaces satisfying Condition A, such that $I-G$ is demiclosed at zero. Under mild assumptions on the control sequences, we establish a strong convergence theorem for the proposed iterative algorithm without requiring prior knowledge of the operator norm. Furthermore, numerical experiments and applications to image restoration problems are provided to demonstrate the efficiency and superiority of the algorithm compared to existing methods.

1. INTRODUCTION

Fixed-point theory serves as a fundamental tool in nonlinear analysis, owing to its extensive utility in resolving complex issues across various scientific domains, such as signal recovery, image processing, and mathematical optimization. These applications can be effectively formulated for finding fixed points within the framework of Hilbert spaces. Among the various classes of operators, nonexpansive and β-demicontractive mappings have emerged as particularly significant due to providing a foundation for developing efficient computational methods.

Herein, let H be a real Hilbert space and C be a nonempty closed convex subset of H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Denote the set of all **fixed-points** of F is denoted by

$$Fix(F) = \{x \in H : Fx = x\}.$$

A mapping $F : H \to H$ is said to be as follows:

(i) *Nonexpansive* if

$$\|Fx - Fy\| \leq \|x - y\|, \qquad \text{for all } \ x, y \in H.$$

(ii) *Quasi-nonexpansive* if $Fix(F) \neq \varnothing$ and

$$\|Fx - y\| \leq \|x - y\|, \qquad \text{for all } \ x \in H \text{ and } \ y \in Fix(F)$$

(iii) *β-demicontractive* if $Fix(F) \neq \varnothing$, and there exists a positive number $\beta < 1$ such that

$$\|Fx - y\|^2 \leq \|x - y\|^2 + \beta\|x - Fy\|^2, \qquad \text{for all } x \in H \text{ and } y \in Fix(F).$$

(iv) *Strongly quasi-nonexpansive* if $Fix(F) \neq \varnothing$, F is quasi-nonexpansive and $x_n - Fx_n \to 0$ whenever $\{x_n\}$ is a bounded sequence such that $\|x_n - x\| - \|Fx_n - x\| \to 0$ for some $x \in Fix(F)$.

Received: 10.08.2025. In revised form: 20.12.2025. Accepted: 17.06.2026
2020 *Mathematics Subject Classification.* 47H09, 47H10, 47J25, 49M37.
Key words and phrases. *Hilbert space, nonexpansive mapping, β-demicontractive mapping, fixed point, strong convergence.*
Corresponding author: Tadchai Yuying; tadchai.yuy@uru.ac.th

Common fixed point problem for mappings F and G is generally denoted by the following:

$$\text{find } x \in H \text{ such that } Fx = Gx = x.$$

To achieve efficient numerical solutions, numerous iterative schemes have been proposed. A cornerstone of these approaches is the well-known Mann-type iteration [14, 15], defined as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Gx_n,$$

where $\{\alpha_n\} \in (0, 1)$ is a relaxation parameter which satisfies some appropriate assumptions. However, Mann iterations often suffer from slow convergence rates. To enhance computational efficiency, researchers have integrated the inertial acceleration technique, a concept in Polyak's heavy ball method [16]. By incorporating a term based on the difference between previous iterations, this technique significantly boosts the convergence speed of the algorithm. Polyak's heavy ball method [16] has been presented as follows:

$$x_{n+1} = x_n + \theta_n(x_n + x_{n-1}),$$

where $\theta_n(x_n + x_{n-1})$ is inertial term. Many researchers are interested in adding an initial technique to the algorithm to improve the efficiency of the algorithms they have developed, see more [1, 2, 6, 10] the references therein.

In other words, research concerning common fixed-point problems remains a growing field, driven by its broad applicability in solving complex problems within various scientific domains. In 2009, Iemoto and Takahashi [12] approximated the common fixed points of a nonexpansive mapping $T : H \to H$ and nonspreading mapping $S : H \to H$ in Hilbert space as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[(1 - \beta_n)Fx_n + \beta_n Gx_n],$$

where $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ are a relaxation parameter.

Recently, Berinde and Saleh [4] present Halpern-type algorithm for approximating the solution of a common fixed-point for a couple of nonexpansive and β-demicontractive mappings where $\beta < 1$ with a variational inequality constraint as follows:

$$x_{n+1} = a_n u + (1 - a_n)[b_n Fx_n + (1 - b_n)((1 - \alpha)x_n + \alpha Gx_n)], \tag{1.1}$$

where u is arbitrary in H, $\{a_n\}, \{b_n\} \in (0, 1)$ and $\alpha \in (0, 1 - \beta)$ are a relaxation parameter. Many researchers adapted the fixed point to common fixed point problems to increase the diversity and dimension of the problem, see for example [4, 7, 8, 12, 17, 20, 21] and the references therein. One of the famous applications for the iteration algorithm of fixed point and common fixed point problems is applying the algorithm to image restoration problems. That is, applying the fixed point problem to the monotone inclusion problem [3, 22, 23]. Furthermore, a common problem in deploying these algorithms is the dependence on the operator norm to determine the step size. In calculating the exact norm is computationally expensive or even impossible. These reasons for the development of self-adaptive step size, which allow the algorithm to dynamically update parameters without requiring prior information about the operator's norm.

Drawing inspiration from these developments, this research proposes an averaged Mann-type algorithm that synergistically combines inertial acceleration with a self-adaptive step size mechanism. Our study focuses on approximating common fixed points for nonexpansive and β-demicontractive mappings. We prove a strong convergence theorem for the proposed scheme in a real Hilbert space under mild conditions. Finally, the practical efficacy of our method is validated through numerical simulations and its application to image restoration, demonstrating superior performance over existing iterative frameworks.

2. PRELIMINARIES

We will use the following notation:
(1) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
(2) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Now, we present some fact and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1. *Let H be a Hilbert space. Then the following results hold:*

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$;
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(iii) *for all* $t \in [0, 1]$ *and* $x, y \in H$,

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2.$$

Lemma 2.2 ([5]). *Let H be a Hilbert space and $D \subset H$ be a closed and convex set. If $G : D \to D$ is β-demicontractive, then the Krasnoselskij perturbation $G_\delta = (1 - \delta)I + \delta G$ of G is $(1 + \beta/\delta - 1/\delta)$-demicontractive.*

Lemma 2.3 ([5]). *Let H be a Hilbert space and $D \subset H$ be a closed and convex set. If $G : D \to D$ is β-demicontractive, then for any $\delta \in (0, 1 - \beta)$,*

$$G_\delta = (1 - \delta)I + \delta G$$

is quasi-nonexpansive.

Definition 2.4. A mapping $T : D \to D$ is said to satisfy **Condition (A)** if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in D$, where $d(x, F(T))$ denotes the distance from x to the fixed point set $F(T)$.

Example 2.5. Let $H = \mathbb{R}$ and $D = [0, 1]$. Define $Tx = \dfrac{x}{4}$ for all $x \in D$. It is easy to see that $F(T) = \{0\}$. For any $x \in D$, we have $\|x - Tx\| = \dfrac{3}{4}|x|$. If we define a non-decreasing function $f : [0, \infty) \to [0, \infty)$ by $f(r) = \dfrac{1}{2}r$, then $f(0) = 0$ and $f(r) > 0$ for $r > 0$. Since $d(x, F(T)) = |x|$, it follows that $\|x - Tx\| = \dfrac{3}{4}|x| \geq \dfrac{1}{2}|x| = f(d(x, F(T)))$. Thus, T satisfies Condition (A).

Lemma 2.6 ([19]). *Let $\{c_n\}$ be a sequence of nonnegative real numbers and $\{d_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} t_n = \infty$. Assume that*

$$c_{n+1} \leq (1 - t_n)c_n + t_n d_n$$

for all $n \in \mathbb{N}$. If $\limsup i \to \infty d_{n_i} \leq 0$ for every subsequence $\{c_{n_i}\}$ of $\{c_n\}$ satisfying

$$\liminf_{i \to \infty}(c_{n_{i+1}} - c_n) \geq 0.$$

Then $\lim_{n \to \infty} c_n = 0$.

Lemma 2.7 ([24]). *Let C be a nonempty subset of a real Hilbert space and let $T : C \to C$ be a k-strictly pseudocontractive mapping. Then, the averaged mapping $T_\lambda = (1 - \lambda)I + \lambda T$ is nonexpansive for any $\lambda \in (0, 1 - k)$.*

Let us now present Mann-type algorithm with inertial technique as follows:

Algorithm 1.

(1) **Initial.** Take $x_0, x_1 \in H$ arbitrarily and $s_1 = 0$

(2) **For** $n \geq 1$, we set

$$\theta_n = \begin{cases} \min\left\{\frac{s_n-1}{s_{n+1}}, \frac{\alpha_n b_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \frac{s_n-1}{s_{n+1}}, & \text{otherwise;} \end{cases} \tag{2.1}$$

where $s_{n+1} = \frac{1+\sqrt{1+4s_n^2}}{2}$ and $\{\alpha_n\} \subset [0,1)$.

(3) **Step 1**. Calculate u_n, v_n and x_{n+1} using:

$$\begin{aligned} u_n &= x_n + \theta_n(x_n - x_{n-1}) \\ v_n &= b_n F u_n + (1-b_n)((1-\gamma)u_n + \gamma G u_n) \\ x_{n+1} &= a_n F v_n + (1-a_n)v_n, \end{aligned} \tag{2.2}$$

where $\{a_n\}, \{b_n\} \subset [0,1]$.

(4) If $\|x_{n+1} - x_n\| < \varepsilon$, stop. Otherwise, set $n := n+1$ and return to Step 1.

Algorithm 2.

(1) **Initial.** Take $x_0, x_1 \in H$ arbitrarily and $s_1 = 0$

(2) **For** $n \geq 1$, we set

$$\theta_n = \begin{cases} \min\left\{\frac{s_{n-1}}{s_{n+1}}, \frac{\alpha_n b_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \frac{s_n-1}{s_{n+1}}, & \text{otherwise;} \end{cases} \tag{2.3}$$

where $s_{n+1} = \frac{1+\sqrt{1+4s_n^2}}{2}$ and $\{\alpha_n\} \subset [0,1)$.

(3) **Step 1**. Calculate u_n, v_n and x_{n+1} using:

$$\begin{aligned} u_n &= x_n + \theta_n(x_n - x_{n-1}) \\ v_n &= b_n \Gamma_\lambda u_n + (1-b_n)((1-\gamma)u_n + \gamma B u_n) \\ x_{n+1} &= a_n B v_n + (1-a_n)v_n, \end{aligned} \tag{2.4}$$

where $\Gamma_\lambda u = (1-\lambda)u + \lambda \Gamma u$, with $\lambda \in (0, 1-k)$ and $\{a_n\}, \{b_n\} \subset [0,1]$.

(4) If $\|x_{n+1} - x_n\| < \varepsilon$, stop. Otherwise, set $n := n+1$ and return to Step 1.

3. Common Fixed Points

We first introduce the following property to help prove the main results.

A mapping G satisfies **Condition A** if $x_n - Gx_n \to 0$ whenever $\{x_n\}$ is bounded sequence such that

$$\|x_n - x^*\| - \|(1-\delta)x_n + \delta G x_n - x^*\| \to 0$$

for some $x^* \in Fix(G)$ and $\delta \in [0,1]$.

Theorem 3.1. *Let H be a Hilbert space and C be a closed convex subset of H, A mappings $F : C \to C$ is a nonexpansive and $G : C \to C$ is a β-demicontractive satisfying Condition A such that $I - G$ is demiclosed at 0. Assume that $Fix(F) \cap Fix(G) \neq \varnothing$. Let $\{a_n\}$ and $\{b_n\}$ be the sequences in $[0,1]$ which satisfy the following conditions:*

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$,
(ii) $\sum_{n=1}^{\infty} b_n = \infty$,
(iii) $a_n \in (a,b)$ *for some* $a, b \in (0,1)$ *with* $a < b$,
(iv) $\lim_{n\to\infty} \alpha_n \to 0$.

Let $x_0, x_1 \in H$ be arbitrarily and $\{x_n\}$ be a sequence generated by Algorithm 1. Then $\{x_n\}$ converges strongly to $x \in Fix(F) \cap Fix(G)$ where $x^* = P_{Fix(F)\cap Fix(G)}(x)$.

Proof. We first to show that $\{x_n\}$ is a bounded sequence. By the definition of u_n and v_n, we get

$$\|u_n - x\| = \|x_n + \theta_n(x_n - x_{n-1}) - x\|$$

$$\leq \|x_n - x\| + \theta_n\|x_n - x_{n-1}\|. \tag{3.1}$$

Since G is a β-demicontractive mapping, by Lemma 2.3 and follows the averaged mapping $G_\gamma = (1-\gamma)I + \gamma G$ is quasi-nonexpansive for $\gamma \in (0, 1-\beta)$. Clearly, $G_\gamma - I$ is demiclosed at zero. One can also see that G_γ is strongly quasi-nonexpansive from the fact that G satisfies Condition A. Now, we can write

$$v_n = b_n F u_n + (1-b_n)G_\gamma u_n,$$

for each $n \in \mathbb{N}$. Let x be a common fixed point of F and G_γ. Define

$$T_n = b_n F + (1-b_n)G_\gamma,$$

for each $n \in \mathbb{N}$. So, we get

$$\begin{aligned} \|v_n - x\| &= \|T_n u_n - x\| \\ &\leq \|u_n - x\| \\ &\leq \|x_n - x\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.2}$$

Since (3.1), (3.2), we get

$$\begin{aligned} \|x_{n+1} - x\| &= \|a_n(Fv_n - Fx) + a_n(Fx - x) + (1-a_n)(v_n - x)\| \\ &\leq a_n\|Fv_n - Fx\| + a_n\|Fx - x\| + (1-a_n)\|v_n - x\| \\ &\leq a_n\|v_n - x\| + a_n\|Fx - x\| + (1-a_n)\|v_n - x\| \\ &\leq \|v_n - x\| + a_n\|Fx - x\| \leq \|x_n - x\| + \theta_n\|x_n - x_{n-1}\| + a_n\|Fx - x\|. \end{aligned} \tag{3.3}$$

This implies that

$$\|x_{n+1} - x\| \leq \|x_n - x\| + a_n\Big(\frac{\theta_n}{a_n}\|x_n - x_{n-1}\| + \|Fx - x\|\Big)$$

From $\dfrac{\theta_n}{a_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$, there exists a constant $M > 0$ such that $\dfrac{\theta_n}{a_n}\|x_n - x_{n-1}\| \leq M$, for all $n \geq 1$. Therefore,

$$\|x_{n+1} - x\| \leq \|x_n - x\| + a_n(M + \|Fx - x\|). \tag{3.4}$$

By mathematical induction, we obtain

$$\|x_{n+1} - x\| \leq \max\{\|x_0 - x\|, M + \|Fx - x\|\},$$

for all $n \geq 1$. This implies that $\{x_n\}$ is bounded. By definition of w_n, for each $n \in \mathbb{N}$, we get

$$\begin{aligned} \|u_n - x\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x\|^2 \\ &= \|x_n - x\|^2 + 2\theta_n\langle x_n - x, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.5}$$

Using Lemma 2.1(i)-(iii), F is nonexpansive and G_γ is quasi-nonexpansive, we get

$$\begin{aligned} \|v_n - x\|^2 &= \|b_n F u_n + (1-b_n)G_\gamma u_n - x\|^2 \\ &= \|b_n(Fu_n - Fx) + b_n(Fx - x) + (1-b_n)(G_\gamma u_n - x)\|^2 \\ &\leq \|b_n(Fu_n - Fx) + (1-b_n)(G_\gamma u_n - x)\|^2 + 2\langle b_n(Fx - x), v_n - x\rangle \\ &= b_n\|Fu_n - Fx\|^2 + (1-b_n)\|G_\gamma u_n - x\|^2 \\ &\quad - b_n(1-b_n)\|(Fu_n - Fx) - (G_\gamma u_n - x)\|^2 + 2\langle b_n(Fx - x), v_n - x\rangle \\ &= b_n\|Fu_n - Fx\|^2 + (1-b_n)\|G_\gamma u_n - x\|^2 \\ &\quad - b_n(1-b_n)[\|Fu_n - Fx\|^2 - 2\langle Fu_n - Fx, G_\gamma u_n - x\rangle + \|G_\gamma u_n - x\|^2] \\ &\quad + 2\langle b_n(Fx - x), v_n - x\rangle \end{aligned}$$

$$
\begin{aligned}
&\leq b_n\|Fu_n - Fx\|^2 + (1-b_n)\|G_\gamma u_n - x\|^2 \\
&\quad - b_n(1-b_n)[\|Fu_n - Fx\|^2 - 2\|Fu_n - Fx\|\|G_\gamma u_n - x\| + \|G_\gamma u_n - x\|^2] \\
&\quad + 2\langle b_n(Fx - x), v_n - x\rangle \\
&\leq b_n^2\|Fu_n - Fx\|^2 + (1-b_n)^2\|G_\gamma u_n - x\|^2 \\
&\quad + 2b_n(1-b_n)\|Fu_n - Fx\|\|G_\gamma u_n - x\| + 2\langle b_n(Fx - x), v_n - x\rangle \\
&\leq b_n^2\|u_n - x\|^2 + (1-b_n)^2\|u_n - x\|^2 \\
&\quad + 2b_n(1-b_n)\|u_n - x\|\|u_n - x\| + 2\langle b_n(Fx - x), v_n - x\rangle \\
&\leq (b_n^2 + (1-b_n)^2 + 2b_n(1-b_n))\|u_n - x\|^2 + 2b_n\langle Fx - x, v_n - x\rangle \\
&= (1-4b_n)\|u_n - x\|^2 + 2b_n\langle Fx - x, v_n - x\rangle \\
&= (1-4b_n)(\|x_n - x\|^2 + 2\langle x_n - x, \theta_n(x_n - x_{n-1})\rangle + \theta_n^2\|x_n - x_{n-1}\|^2) \\
&\quad + 2b_n\langle Fx - x, v_n - x\rangle. \qquad (3.6)
\end{aligned}
$$

Using Lemma 2.1(iii), (3.6), this implies that

$$
\begin{aligned}
\|x_{n+1} - x\|^2 &= \|a_nFv_n + (1-a_n)v_n - x\|^2 \\
&= \|a_n(Fv_n - x) + (1-a_n)(v_n - x)\|^2 \\
&= a_n\|Fv_n - x\|^2 + (1-a_n)\|v_n - x\|^2 - a_n(1-a_n)\|Fv_n - v_n\|^2 \\
&\leq \|v_n - x\|^2 - a_n(1-a_n)\|Fv_n - v_n\|^2 \\
&= (1-4b_n)(\|x_n - x\|^2 + 2\langle x_n - x, \theta_n(x_n - x_{n-1})\rangle + \theta_n^2\|x_n - x_{n-1}\|^2) \\
&\quad + 2b_n\langle Fx - x, v_n - x\rangle - a_n(1-a_n)\|Fv_n - v_n\|^2 \\
&\leq (1-b_n)\|x_n - x\|^2 - a_n(1-a_n)\|Fv_n - v_n\|^2 + b_nd_n, \qquad (3.7)
\end{aligned}
$$

where

$$
d_n = \Big\{2\langle Fx - x, v_n - x\rangle + 2\|x_n - x\|\Big(\frac{\theta_n}{b_n}\|x_n - x_{n-1}\|\Big) + \Big(\frac{\theta_n}{b_n}\|x_n - x_{n-1}\|\Big)\theta_n\|x_n - x_{n-1}\Big\}.
$$

It follows that

$$
a_n(1-a_n)\|Fv_n - v_n\|^2 \leq (1-b_n)\|x_n - x\|^2 - \|x_{n+1} - x\|^2 + b_nM', \qquad (3.8)
$$

where $M' = \sup\{d_n : n \in \mathbb{N}\}$. We finally to show that $x_n \to x$ as $n \to \infty$. To do this, we will apply Lemma 2.6 by setting $c_n := \|x_n - x\|^2$ and $t_n := b_n$. From (3.7), we get the following inequality:

$$
c_{n+1} \leq (1-t_n)c_n + t_nd_n.
$$

Suppose that $\{c_{n_i}\}$ is a subsequence of $\{c_n\}$ such that

$$
\liminf_{i\to\infty}(a_{n_{i+1}} - a_{n_i}) \geq 0. \qquad (3.9)
$$

From (3.8), (3.9) and $a_n \to \infty$ as $n \to \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$, we get

$$
\begin{aligned}
\limsup_{i\to\infty} a_{n_i}(1-a_{n_i})\|Fv_{n_i} - v_{n_i}\|^2 &\leq \limsup_{i\to\infty}(a_{n_i} - a_{n_{i+1}} + M') \\
&\leq \limsup_{i\to\infty}(a_{n_i} - a_{n_{i+1}}) \\
&= -\liminf_{i\to\infty}(a_{n_{i+1}} - a_{n_i}) \\
&\leq 0.
\end{aligned}
$$

From the condition (ii) and above inequality lead to

$$
\lim_{i\to\infty}\|Fv_{n_i} - v_{n_i}\| = 0. \qquad (3.10)
$$

By the condition of θ_n (2.1) together with $\lim_{n\to\infty}\alpha_n = 0$. We note that $\frac{\theta_n}{b_n}\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$, we next show that $\limsup_{i\to\infty} d_{n_i} \leq 0$. Obviously, it suffices to show that

$$\limsup_{n\to\infty}\langle F(x)-x, v_{n_i}-x\rangle < 0.$$

Let $\{v_{n_{i_j}}\}$ be subsequence of $\{v_{n_i}\}$ such that

$$\lim_{j\to\infty}\langle F(x)-x, v_{n_{i_j}}-x\rangle = \limsup_{i\to\infty}\langle F(x)-x, v_{n_i}-x\rangle$$

and $v_{n_{i_j}} \rightharpoonup w$ for some $w \in H$. By (3.10) and set the equation $x = P_{Fix(F)\cap Fix(G)}(x)$ yields

$$\begin{aligned}\limsup_{i\to\infty}\langle F(x)-x, v_{n_i}-x\rangle &= \lim_{j\to\infty}\langle F(x)-x, v_{n_{i_j}}-x\rangle \\ &= \langle F(x)-x, w-x\rangle \leq 0.\end{aligned}$$

Using Lemma 2.6, we can conclude that $x_n \to x$ as $n \to \infty$. □

Remark 3.2. The integration of the inertial technique $\theta_n(x_n - x_{n-1})$ into the averaged Mann-type iterative scheme plays a crucial role in accelerating the convergence process. This inertial term, often referred to as the momentum effect, utilizes the information from the previous two iterates to determine the next position, thereby significantly reducing the number of iterations required to reach the desired accuracy compared to the non-inertial version presented in [4].

Corollary 3.3. *Let H be a Hilbert space and C be a closed convex subset of H, A mappings $F : C \to C$ is a nonexpansive and $G : C \to C$ is a β-demicontractive satisfying Condition A such that $I - G$ is demiclosed at 0. Assume that $Fix(F) \cap Fix(G) \neq \varnothing$. Let $\{a_n\}, \{b_n\}$ be the sequences in $[0, 1]$ and $\{x_n\}$ be the sequence generated as follows:*

$$\begin{aligned}&x_1 \in C,\\ &v_n = b_nFx_n + (1-b_n)[(1-\gamma)x_n + \gamma Gx_n],\\ &x_{n+1} = a_nFv_n + (1-a_n)v_n,\end{aligned}$$

for $n \geq 1$ which satisfy the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$,
(ii) $\sum_{n=1}^{\infty} b_n = \infty$,
(iii) $a_n \in (a, b)$ *for some* $a, b \in (0, 1)$ *with* $a < b$.

Then $\{x_n\}$ converges strongly to $x \in Fix(F) \cap Fix(G)$ where $x^* = P_{Fix(F)\cap Fix(G)}(x)$.

Proof. If $\theta_n = 0$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem (3.1) then $u_n = x_n$. And the rest of the proof is similar to that of Theorem (3.1), we obtain $\{x_n\}$ converges strongly to $x \in Fix(F) \cap Fix(G)$. □

If we set $\theta_n = 0$ and $Fv_n = u$, we obtain the following results of Berinde and Saleh [4].

Corollary 3.4 ([4])**.** *Let H be a Hilbert space and C be a closed convex subset of H, A mappings $F : C \to C$ is a nonexpansive and $G : C \to C$ is a β-demicontractive satisfying Condition A such that $I - G$ is demiclosed at 0. Assume that $Fix(F) \cap Fix(G) \neq \varnothing$. Let $\{a_n\}, \{b_n\}$ be the sequences in $[0, 1]$ and $\{x_n\}$ be the sequence generated as follows:*

$$x_{n+1} = a_nu + (1-a_n)[b_nFx_n + (1-b_n)((1-\alpha)x_n + \alpha Gx_n)],$$

where u is arbitrary in H. If $\sum_{n=1}^{\infty}(1-b_n) < \infty$ and $\frac{b_n}{a_n} \to 0$. then $\{x_n\}$ converges strongly to $x \in Fix(F) \cap Fix(G)$ which is the unique point in $Fix(F) \cap Fix(G)$ that solves the variaional inequality

$$\langle u^* - x, u - x\rangle \geq 0, \quad \forall u \in Fix(F) \cap Fix(G),$$

i.e., $v = P_{Fix(F)\cap Fix(G)}(x)$.

Next, more general theorems can be proven similarly to the proof of Theorem 3.1.

Theorem 3.5. *Let H be a Hilbert space and C be a closed convex subset of H, A mappings $\Gamma : C \to C$ is a k-strictly pseudocontractive and $B : C \to C$ is a β-demicontractive satisfying Condition A such that $I - B$ is demiclosed at 0. Assume that $Fix(\Gamma) \cap Fix(B) \neq \varnothing$. Let $\{a_n\}$ and $\{b_n\}$ be the sequences in $[0, 1]$ which satisfy the following conditions:*

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$,
(ii) $\sum_{n=1}^{\infty} b_n = \infty$,
(iii) $a_n \in (a, b)$ *for some* $a, b \in (0, 1)$ *with* $a < b$,
(iv) $\lim_{n\to\infty} \alpha_n \to 0$.

Let $x_0, x_1 \in H$ be arbitrarily and $\{x_n\}$ be a sequence generated by Algorithm 2. Then $\{x_n\}$ converges strongly to $x \in Fix(\Gamma) \cap Fix(B)$ where $x^* = P_{Fix(\Gamma)\cap Fix(B)}(x)$.

Proof. Since Γ is k-strictly pseudocontractive, by Lemma 2.7, we have that the averaged mapping

$$\Gamma_\lambda = (1 - \lambda)I + \lambda\Gamma$$

is nonexpansive for all $\lambda \in (0, 1 - k)$ and $Fix(\Gamma) = Fix(\Gamma_\lambda)$. Therefore, Γ_λ and B satisfy the conditions of the operators in Theorem 3.1. Now, the rest of the proof is similar to that of Theorem 3.1 by replacing F and G_γ in Theorem 3.1 by Γ_λ and B, respectively. □

From Theorem 3.5, we obtain the following results

Corollary 3.6. *Let H be a Hilbert space and C be a closed convex subset of H, A mappings $\Gamma : C \to C$ is k-strictly pseudocontractive and $B : C \to C$ is a β-demicontractive satisfying Condition A such that $I - B$ is demiclosed at 0. Assume that $Fix(\Gamma_\lambda) \cap Fix(B) \neq \varnothing$. Let $\{a_n\}, \{b_n\}$ be the sequences in $[0, 1]$ and $\{x_n\}$ be the sequence generated as follows:*

$$\begin{aligned} &x_1 \in C, \\ &v_n = b_n\Gamma_\lambda x_n + (1 - b_n)[(1 - \gamma)x_n + \gamma Bx_n], \\ &x_{n+1} = a_n\Gamma_\lambda v_n + (1 - a_n)v_n, \end{aligned}$$

for $n \geq 1$ which satisfy the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$,
(ii) $\sum_{n=1}^{\infty} b_n = \infty$,
(iii) $a_n \in (a, b)$ *for some* $a, b \in (0, 1)$ *with* $a < b$,

Then $\{x_n\}$ converges strongly to $x \in Fix(\Gamma_\lambda) \cap Fix(B)$ where $x^* = P_{Fix(\Gamma_\lambda)\cap Fix(B)}(x)$.

Proof. If $\theta_n = 0$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem (3.1) then $u_n = x_n$. And the rest of the proof is similar to that of Theorem (3.5), we obtain $\{x_n\}$ converges strongly to $x \in Fix(\Gamma_\lambda) \cap Fix(B)$. □

If we set $\theta_n = 0$ and $Fv_n = u$, we obtain the following results of Berinde and Saleh [4].

Corollary 3.7 ([4])**.** *Let H be a Hilbert space and C be a closed convex subset of H, A mappings $F : C \to C$ is k-strictly pseudocontractive and $G : C \to C$ is a β-demicontractive satisfying Condition A such that $I - G$ is demiclosed at 0. Assume that $Fix(F) \cap Fix(G) \neq \varnothing$. Let $\{a_n\}, \{b_n\}$ be the sequences in $[0, 1]$ and $\{x_n\}$ be the sequence generated as follows:*

$$x_{n+1} = a_n u + (1 - a_n)[b_n Fx_n + (1 - b_n)((1 - \alpha)x_n + \alpha Gx_n)],$$

where u is arbitrary in H. If $\sum_{n=1}^{\infty}(1 - b_n) < \infty$ and $\dfrac{b_n}{a_n} \to 0$. then $\{x_n\}$ converges strongly to $x \in Fix(F) \cap Fix(G)$ which is the unique point in $Fix(F) \cap Fix(G)$ that solves the variaional inequality

$$\langle u^* - x, u - x\rangle \geq 0, \quad \forall u \in Fix(F) \cap Fix(G),$$

i.e., $v = P_{Fix(F)\cap Fix(G)}(x)$.

4. APPLICATIONS

From our main algorithm, we will present an exploration of the potential and expect the results for specific cases of Algorithm 1 for the resolution of monotone inclusion problems.

We recommend the operator $U : H \to H$ is said to be as follows:

(i) **monotone** if it satisfies $\langle Ux - Uy, x - y\rangle \geq 0$ for all $x, y \in H$.
(ii) κ**-cocoercive** if there exists $\kappa > 0$ such that $\langle Ux - Uy, x - y\rangle \geq \kappa\|Ux - Uy\|^2$ for all $x, y \in H$.

The set of all zeros of the operator U is denoted by $zer(U) := \{z \in H : Uz = 0\}$. Let $V(S) := \{(x, y) \in H \times H : y \in Sx\}$ be the graph of a multivalued operator $S : H \to 2^H$. Then S is called

(i) **monotone** if for all $x, y \in H, u \in Sx$ and $v \in Sy$ implies that $\langle u - v, x - y\rangle \geq 0$.
(ii) **maximal monotone** if $G(V)$ is not properly contained in any graph of other multivalued monotone operator, that is, if $\hat{S} : H \to 2^H$ is a multivalued monotone operator such that $G(S) \subseteq G(\hat{S}$, then $G(S) = G(\hat{S})$.
(iii) ν**-strongly monotone** if there esists $\lambda > 0$ such that $\langle x - y, u - v\rangle \geq \nu\|x - y\|^2$ for all $(x, u), (y, v) \in G(S)$.

The multivalued operator $J_S : H \to 2^H$ which is defined by

$$J_S(x) = (I + S)^{-1}(x) \quad \text{for all } x \in H$$

is called the **resolvent** operator associated with S. It is well know that if $S : H \to 2^H$ is maximal monotone and $\mu > 0$, then $J_{\mu S}$ is single-valued and firmly nonexpansive mappings. This implies that the **Yosida approximation** S_μ of S is defined by $S_\mu := \frac{1}{\mu}(I - J_{\mu S})$.

Let us consider the following monotone inclusion problem:

$$\text{find } x \in H \text{ such that } 0 \in (\sigma_1 Rx + \varphi Sx + \psi_1 Ux) \cap (\sigma_2 Rx + \varphi Sx + \psi_2 Ux) \tag{4.1}$$

where $\sigma_1, \sigma_2, \varphi, \psi_1, \psi_2 > 0, R, S : H \to 2^H$ are maximal monotone operators and $U : H \to H$ is ν-cocoercive operator with $\nu > 0$. We next present the important tool for applying Algorithm 1 to solving (4.1) as follows:

Proposition 4.1 ([9]). *Let $T_1, T_2 : H \to H$ be two firmly nonexpansive operators and W be a ν-cocoercive operator with $\nu > 0$. Let $\psi \in (0, 2\nu)$. Then operator*

$$T := T_1 \circ (2T_2 - I - \psi W \circ T_2) + I - T_2$$

is ς-averaged with coefficient $\varsigma := \dfrac{2\nu}{4\nu - \psi} < 1$. In particular, the following inequality holds for all $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{(1 - \varsigma)}{\varsigma}\|(I - T)x - (I - T)y\|^2.$$

Lemma 4.2 ([22], Fixed point encoding). *Let $\sigma, \varphi, \psi > 0, R, S : H \to 2^H$ be maximal monotone operators and $U : H \to H$ be an operator. Suppose that $zer(\sigma R + \varphi S + \psi U) \neq \varnothing$. Then,*

$$zer(\sigma R + \varphi S + \psi U) = J_{\mu S}(Fix(T)),$$

where $T = J_{\mu R} \circ (2J_{\varphi S} - I - \psi U \circ J_{\varphi S}) + (I - J_{\varphi S})$.

From Proposition4.1 and Lemma 4.2, we can setting the our algorithm as follows:

$$F := J_{\mu_1 R} \circ (2J_{\varphi S} - I - \psi_1 U \circ J_{\varphi S}) + (I - J_{\varphi S})$$

and

$$G := J_{\mu_2 R} \circ (2J_{\varphi S} - I - \psi_2 U \circ J_{\varphi S}) + (I - J_{\varphi S}).$$

This results in the following algorithms for monotone inclusion problem:

Algorithm 3.

(1) **Initial.** Take $x_0, x_1 \in H$ arbitrarily and $s_1 = 0$

(2) **For** $n \geq 1$, we set

$$\theta_n = \begin{cases} \min\left\{\frac{s_n - 1}{s_{n+1}}, \frac{\alpha_n b_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \frac{s_n - 1}{s_{n+1}}, & \text{otherwise;} \end{cases} \tag{4.2}$$

where $s_{n+1} = \frac{1+\sqrt{1+4s_n^2}}{2}$ and $\{\alpha_n\} \subset [0, 1)$.

(3) **Step 1**. Calculate the iterative step:

$$\begin{aligned} u_n &= x_n + \theta_n(x_n - x_{n-1}) \\ v_n &= J_{\varphi S}(u_n) \\ p_n &= J_{\sigma_1 R}(2v_n - u_n - \psi_1 U v_n) + u_n - v_n \\ q_n &= J_{\sigma_2 R}(2v_n - u_n - \psi_2 U v_n) + u_n - v_n \\ z_n &= b_n p_n + (1 - b_n)((1 - \gamma)u_n + \gamma q_n) \\ y_n &= J_{\sigma_1 R}(2v_n - z_n - \psi_1 U v_n) + z_n - v_n \\ x_{n+1} &= a_n y_n + (1 - a_n) z_n, \end{aligned} \tag{4.3}$$

where $\sigma_1, \sigma_2, \varphi > 0$ and $\psi_1, \psi_2 \in (0, 2\nu)$.

Then, update $n := n + 1$ and return to iterative step.

If we set $Sx = 0$, then $J_{\varphi S}(x) = I(x)$ for all $x \in H$. So, Lemma 4.2 reduces to $zer(\sigma_1 R + \psi_1 U) \cap zer(\sigma_2 R + \psi_2 U) = Fix(F) \cap Fix(G)$ where $F := J_{\sigma_1 R} \circ (I - \psi_1 U)$ and $G := J_{\sigma_2 R} \circ (I - \psi_2 U)$. This results in the following algorithm:

Algorithm 4.

(1) **Initial.** Take $x_0, x_1 \in H$ arbitrarily and $s_1 = 0$

(2) **For** $n \geq 1$, we set

$$\theta_n = \begin{cases} \min\left\{\frac{s_n - 1}{s_{n+1}}, \frac{\alpha_n b_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}; \\ \frac{s_n - 1}{s_{n+1}}, & \text{otherwise;} \end{cases} \tag{4.4}$$

where $s_{n+1} = \frac{1+\sqrt{1+4s_n^2}}{2}$ and $\{\alpha_n\} \subset [0, 1)$.

(3) **Step 1**. Calculate the iterative step:

$$\begin{aligned} u_n &= x_n + \theta_n(x_n - x_{n-1}) \\ p_n &= J_{\sigma_1 R}(u_n - \psi_1 U u_n) \\ q_n &= J_{\sigma_2 R}(u_n - \psi_2 U u_n) \\ z_n &= b_n p_n + (1 - b_n)((1 - \gamma)u_n + \gamma q_n) \\ y_n &= J_{\sigma_1 R}(v_n - \psi_1 U v_n) \\ x_{n+1} &= a_n y_n + (1 - a_n) z_n, \end{aligned} \tag{4.5}$$

where $\sigma_1, \sigma_2, \varphi > 0$ and $\psi_1, \psi_2 \in (0, 2\nu)$.

Then, update $n := n + 1$ and return to iterative step.

5. Numerical Experiments

In this section, we introduce some numerical examples to show the performance for comparing Algorithm (1.1) [4] of Berinde and Saleh with Algorithm 1 for a common fixed point of nonexpansive and β-demicontractive mappings and demonstrate image restoration problems for Algorithm 4 by applying real-world scenarios. For computational results, we used the MATLAB tool in a computing environment on a system equipped with a MacBook Pro (13-inch, 2019, with two Thunderbolt 3 ports and 8 GB of 2133 MHz LPDDR3 memory).

5.1. **Numerical illustrations.** For this part, In the paper [13] presented the example of nonexpansive mapping as follows: let $C = [-1, 1]$ and defined $F : C \to C$ by

$$Fx = -2\sin(\frac{x}{2}) \quad \text{where} \quad x \in [0, 1],$$

and [18] introduced the example for $\frac{1}{2}$-demicontractive mappings as follows: let $\mathbb{R}$ denote the reals with the usual norm and $G : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$Gx = \begin{cases} x, & \text{if } -\infty < x < 0; \\ -3x, & \text{if } 0 \leq x < \infty. \end{cases}$$

It is worth noting that the mapping G defined above satisfies Condition (A), ensuring the strong convergence of the proposed algorithm as established in Theorem 3.1.

By Algorithm (1.1) [4], we set $a_n = \dfrac{1}{1000n}, b_n = \dfrac{2n}{1+3p}$ and $\alpha = 0.3$.

In Algorithm 1, we set $s_1 = 0, \gamma = 0.3, \alpha_n = \dfrac{88 \cdot 10^{20}}{n}, a_n = \dfrac{0.9(n+1)}{n}$ and $b_n = \dfrac{1}{88n}$. In the table, Denote Iter. and Sec. are the number of iterations and the CPU times in seconds, respectively. Denote $E(x) = \|x_n - x_{n-1}\|$ and we took $E(x) < \varepsilon$ as the stopping criterion and $\varepsilon = 10^{-4}$. From the setting arbitrary points, the parameters and stopping criterion above, we obtain the result as follows:

Table 1. Comparison of Algorithm (1.1) [4] and Algorithm 1.

inertial		Algorithm (1.1)		Algorithm 1	
$x_0(u)$	x_1	Iter.	Sec.	Iter.	Sec.
0.2	0.9	41570	0.043159	21	0.022803
0.85	0.22	13826	0.032322	10	0.023544
0.247	0.135	38973	0.056078	18	0.022837
0.9843	0.1145	16517	0.035322	14	0.013842

Table 2. Comparison of Algorithm (1.1) [4] and Algorithm 1.

	Algorithm (1.1)		Algorithm 1	
$x_0(u) = x_1$	Iter.	Sec.	Iter.	Sec.
0.6	21633	0.028811	18	0.023049
0.28	7471	0.009315	14	0.004448
0.567	54513	0.05844	18	0.015024
0.3198	2469	0.019252	14	0.067060

5.2. **Image restoration problems.** To further validate the practical utility of the strong convergence results presented in Section 3, we apply Algorithm 1 to the image restoration problem. This problem is formulated as a minimization of a functional that can be equivalently characterized as finding a common fixed point of associated proximal operators.

Recall that the image restoration for the most relevant mathematical approximations can be expressed as:

$$y = \prod x + w, \tag{5.1}$$

where $y, \prod, x$ and w are the corrupted image, blurring operator, pristine image and noise operator, respectively.

To solve the subsequent regularized least-squares problem as follows:

$$\min\{\frac{1}{2}\| \prod x - y\|_2^2 + \tau\|x\|_1\}, \tag{5.2}$$

where y is the corrupted image and $\prod$ is bounded linear operator. Noted that problem (5.2 is a specific instance of problem (4.1) by setting $R = \partial f(\cdot), S = 0$ and $U = \Delta L(\cdot)$ where $f(x) = \|x\|_1, \tau = 0.001$ and $L(x) = \frac{1}{2}\| \prod x - y\|_2^2$. It is the following fact that $U(x) = \Delta L(x) = \prod^*(\prod x - y)$, where $\prod^*$ is a transpose of $\prod$. Defined the configuration outlined in Algorithm 4 by setting $a_n = \frac{0.1(n+1)}{n}, b_n = 1 - \frac{1}{100n^2}, \gamma = \frac{80n}{100n-9}, s_1 = 0, \alpha_n = 0.99999, \sigma_1 = \sigma_2 = 0.000099, \psi_1 = 0.7, \psi_2 = 0.99$ and Algorithm (1.1) by setting $a_n = \frac{70k-9}{100k}, b_n = 0.99 + \frac{1}{10s}, \alpha = \frac{0.1(s+1)}{s}, \sigma_1 = \sigma_2 = 0.000099, \psi_1 = 0.7, \psi_2 = 0.9$. We assess the quality of the reconstructed image by using the signal to noise ratio (**SNR**) for images which is defined by

$$\mathbf{SNR}(n) = 20\log_{10}\frac{\|x\|_2^2}{\|x - x_n\|_2^2},$$

where x and x_n denote the original and the restored image at iteration n, respectively.

This results in the following figures:



(A) Cat (B) Blur

(C) Algorithm (1.1) [4] (D) Algorithm 4.

FIGURE 1. (a) The original image, (b) the images degraded by motion blur and random noise (c-d) show the reconstructed images obtained from Algorithm (1.1) [4] and Algorithm 4.

As illustrated in Figure 1 and Table 1, the proposed Algorithm 1 demonstrates a faster convergence rate than Algorithm (1.1) in [4]. This superior performance is due to the synergistic effect of the averaged operator and the inertial acceleration. While the averaged technique ensures stability and strong convergence, the inertial momentum allows the iterates to take larger, more effective steps towards the fixed point, especially in the early stages of the iteration.

6. CONCLUSION

In this research, we have introduced a novel averaged Mann-type iterative framework that incorporates both an inertial acceleration term and a self-adaptive step size strategy. The developed algorithm is specifically designed to identify common fixed points for the classes of nonexpansive and β-demicontractive operators within real Hilbert spaces, which is Algorithm 1. We have rigorously established a strong convergence theorem for this scheme under a set of mild conditions.

A key innovation of the proposed method lies in its self-adaptive parameter selection, which eliminates the practical difficulty of estimating the operator norm. The empirical results were obtained through comprehensive numerical simulations and image restoration tasks. From comparing the performance, we found that Algorithm 1 represents more numerical advantages than Algorithm (1.1) as shown in Table 1 and Table 2, respectively. Furthermore, we gain the picture restoration result by applying it to monotone inclusion issues, demonstrating that our technique (Algorithm 4) outperforms Algorithm (1.1) [4]

of Berinde and Saleh in image restoration, as shown in Figure 1. All demonstrate that our approach significantly outperforms several existing methods in terms of convergence speed and computational efficiency.

Acknowledgments

The authors would like to acknowledge the support from their respective institutions: Rangsit University (Suparat Baiya), Naresuan University (Somyot Plubtieng), and Uttaradit Rajabhat University (Tadchai Yuying, Corresponding Author).

References

[1] Alvarez, F.; Attouch, H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.* **9** (2001), no. 1-2, 3–11.

[2] Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. *SIAM J. Optim.* **14** (2004), no. 3, 773–782.

[3] Artsawang, N.; Plubtieng, S.; Bagdasar, O.; Ungchittrakool, K.; Baiya, S.; Thammasiri, P. Inertial Krasnosel'skiĭ-Mann iterative algorithm with step-size parameters involving nonexpansive mappings with applications to solve image restoration problems. *Carpathian J. Math.* **40** (2024), no. 2, 243–261.

[4] Berinde, V.; Saleh, K. An Averaged Halpern-Type Algorithm for Solving Fixed-Point Problems and Variational Inequality Problems. *Axioms* **13** (2024), no. 11, Art. 756.

[5] Berinde, V. Approximating fixed points of demicontractive mappings via the quasi-nonexpansive case. *Carpathian J. Math.* **39** (2023), no. 1, 73–84.

[6] Bot, R. I.; Csetnek, E. R.; Hendrich, C. Inertial Douglas-Rachford splitting for monotone inclusion problems. *Appl. Math. Comput.* **256** (2015), 472–487.

[7] Ceng, L.-C.; Yuan, Q. Composite inertial subgradient extragradient methods for variational inequalities and fixed point problems. *J. Inequal. Appl.* **2019** (2019), Paper No. 274.

[8] Ceng, L.-C.; Yuan, Q. Variational inequalities, variational inclusions and common fixed point problems in Banach spaces. *Filomat* **34** (2020), no. 9, 2939–2951.

[9] Davis, D.; Yin, W. A three-operator splitting scheme and its optimization applications. *Set-Valued Var. Anal.* **25** (2017), no. 4, 829–858.

[10] Dong, Q. L.; Yuan, H. B.; Cho, Y. J.; Rassias, Th. M. Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings. *Optim. Lett.* **12** (2018), no. 1, 87–102.

[11] Halpern, B. Fixed points of nonexpanding maps. *Bull. Amer. Math. Soc.* **73** (1967), no. 6, 957–961.

[12] Iemoto, S.; Takahashi, W. Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space. *Nonlinear Anal.* **71** (2009), no. 12, 2082–2089.

[13] Liu, Z.; Feng, C.; Ume, J. S.; Kang, S. M. Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings. *Taiwanese J. Math.* **11** (2007), no. 1, 27–42.

[14] Mann, W. R. Mean value methods in iteration. *Proc. Amer. Math. Soc.* **4** (1953), no. 3, 506–510.

[15] Krasnosel'skii, M. A. Two remarks on the method of successive approximations. *Russian Math. Surveys* **10** (1955), no. 1, 123–127.

[16] Polyak, B. T. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* **4** (1964), no. 5, 1–17.
[17] Moudafi, A. Krasnoselski-Mann iteration for hierarchical fixed-point problems. *Inverse Problems* **23** (2007), no. 4, 1635–1640.
[18] Mongkolkeha, C.; Cho, Y. J.; Kumam, P. Convergence theorems for k-demicontractive mapping in Hilbert spaces. *Math. Inequal. Appl.* **16** (2013), no. 4, 1065–1082.
[19] Saejung, S.; Yotkaew, P. Approximation of zeros of inverse strongly monotone operators in Banach spaces. *Nonlinear Anal.* **75** (2012), no. 2, 724–750.
[20] Takahashi, W.; Tamura, T. Convergence theorems for a pair of nonexpansive mappings. *J. Convex Anal.* **5** (1998), no. 1, 45–56.
[21] Thongsri, P.; Suantai, S. A new viscosity approximation method with inertial technique for convex bilevel optimization problems and applications. *Carpathian J. Math.* **40** (2024), no. 2, 477–491.
[22] Ungchittrakool, K.; Plubtieng, S.; Artsawang, N.; Thammasiri, P. Modified Mann-type algorithm for two countable families of nonexpansive mappings and application to monotone inclusion and image restoration problems. *Mathematics* **11** (2023), no. 11, Art. 2465.
[23] Ungchittrakool, K.; Artsawang, N. A generalized viscosity forward-backward splitting scheme with inertial terms for solving monotone inclusion problems and its applications. *AIMS Mathematics* **9** (2024), no. 9, 23632–23650.
[24] Zhou, H. Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces. *Nonlinear Anal.* **69** (2008), no. 2, 456–462.

[1]Department of Mathematics and Statistics, Faculty of Science,
Rangsit University, Pathumthani 12000, Thailand
Email address: suparat.b@rsu.ac.th

[2]Department of Mathematics, Faculty of Science, Naresuan University,
Phitsanulok 65000, Thailand
Email address: somyotp@nu.ac.th

[3]Department of Mathematics, Uttaradit Rajabhat University,
Uttaradit 53000, Thailand
Email address: tadchai.yuy@uru.ac.th