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Existence of Solutions for Single-Leader Multi-Follower Games via Direct Preference Maps

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ABSTRACT. This paper is concerned with the existence of a solution for a single-leader multi-follower game (SLMFG), where the followers jointly solve an abstract economy problem. Recall that an abstract economy problem is an extension of a generalized Nash equilibrium problem (GNEP) in the sense that the preference of each player can be described without numerical criteria. The results of this paper, therefore, extend the known literature concerning the existence of a solution of a SLMFG. Due to the lack of criterion functions in the lower-level, the technique we used is quite different from those that deal with GNEPs. In particular, we argue that the follower's abstract economy profiles, *i.e.*, constraint-preference couples, belong to a particular metric space where the response map is proved to be upper semicontinuous.

1. INTRODUCTION

A bilevel programming, or bilevel optimization problem, refers to a class of optimization problems that involve two players, one is classified as the *leader* and the other one as the *follower*. When the leader makes her decision, the follower observes and makes an *optimal response* by solving the *lower-level* optimization problem, parametrized by the leader's action. The goal of the leader is then to make a decision so that, altogether with the follower's optimal response, gives her the *best* objective value. When the single follower is replaced with multiple followers, they respond to the leader's action by solving a parametrized equilibrium-type problem. This approach is broadly referred to as a *single-leader multi-follower game* (or briefly, *SLMFG*). These classes of bilevel problems have posed several interesting and important questions, theoretically and practically, in the literature. For instance, the issues regarding the KKT reformulation (or MPCC reformulation) were discussed in [9, 12, 14]. The existence results for several different settings of a SLMFG were studied in [1, 7, 8, 10, 11].

In this paper, we propose an existence theorem for a single-leader multi-follower game (SLMFG) in which the followers' preferences are directly given by the preference maps without the need of objective functions. Thus, the lower-level problem here is presented as an abstract economy, à la Shafer and Sonnenschein [16], parametrized by the leader's action. Let us state precisely the problem of our study for this paper. Consider the n followers that are represented by a set $\mathtt{F} := \{1, \ldots, n\}$. The single-leader, denoted by ϑ, owns an action set W and each follower $\nu \in \mathtt{F}$ owns an action set X_ν. We write $X := \prod_{\nu\in\mathtt{F}} X_\nu$ and for each $\nu \in \mathtt{F}$ we write $X_{-\nu} := \prod_{\mu\in\mathtt{F}\setminus\{\nu\}} X_\mu$ whose generic element is denoted by $x_{-\nu}$. It is typical to represent X as $X_\nu \times X_{-\nu}$ and to decompose $x \in X$ into $(x_\nu, x_{-\nu}) \in X_\nu \times X_{-\nu}$, and also to extract $x_{-\nu}$ from x the *other followers' component*. The leader is equipped with an objective function $F : W \times X \to \mathbb{R}$, while each follower $\nu \in \mathtt{F}$ is equipped with a constraint map $A_\nu : W \times X_{-\nu} \rightrightarrows X_\nu$ and a preference map P_ν :

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$W \times X \rightrightarrows X_\nu$. For each $w \in W$ and $\nu \in \mathtt{F}$, it is customary to write $A^w_\nu(\cdot) := A_\nu(w,\cdot)$ and $P^w_\nu(\cdot) := P_\nu(w,\cdot)$. The SLMFG that we focus on in this paper has the following form:

$$\begin{cases} \min\limits_{w,x} & F(w,x) & \text{(1.1a)}\\ \text{s.t.} & w \in Y \subset W & \text{(1.1b)}\\ & \forall \nu \in \mathtt{F}: & \text{(1.1c)}\\ & \quad x_\nu \in A^w_\nu(x_{-\nu}), \quad A^w_\nu(x_{-\nu}) \cap P^w_\nu(x) = \varnothing. & \text{(1.1d)} \end{cases}$$

In particular, the constraints (1.1c)–(1.1d) describe that the followers respond to the leader's signal w with a *Nash equilibrium* of a *w-parametrized abstract economy* given by the pair of maps $(A^w_\nu, P^w_\nu)_{\nu\in\mathtt{F}}$. For convenience, at each $w \in W$, we write

$$\mathsf{NE}(w) := \{x = (x_\nu)_{\nu\in\mathtt{F}} \in X \mid \forall \nu \in \mathtt{F}:\ x_\nu \in A^w_\nu(x_{-\nu}), \quad A^w_\nu(x_{-\nu}) \cap P^w_\nu(x) = \varnothing\}$$

to denote the set of all such Nash equilibrium points.

The SLMFG (1.1) indeed covers many particular cases of Nash equilibrium-like problems. For instance, if one fix $\varepsilon \geq 0$ and set

$$P^w_\nu(x_\nu, x_{-\nu}) := \{y_\nu \in X_\nu \mid f_\nu(w, y_\nu, x_{-\nu}) < f_\nu(w, x_\nu, x_{-\nu}) - \varepsilon, \quad (\forall y \in X_\nu)\},$$
$$f_\nu : W \times X \to \mathbb{R},$$

then (1.1) reduces to

$$\begin{cases} \min\limits_{w,x} & F(w,x) & \text{(1.2a)}\\ \text{s.t.} & w \in Y \subset W & \text{(1.2b)}\\ & \forall \nu \in \mathtt{F}: & \text{(1.2c)}\\ & \quad x_\nu \in \varepsilon - \arg\min\limits_{y_\nu}\{f_\nu(w, y_\nu, x_{-\nu}) \mid y_\nu \in A^w_\nu(x_{-\nu})\}, & \text{(1.2d)} \end{cases}$$

which is a single-leader multi-follower game where the followers solve for an ε-Nash equilibrium of a w-parametrized (generalized) Nash equilibrium problem. When $\varepsilon = 0$, this is a classical single-leader multi-follower game; you can see more here [17, 13].

Likewise, one could also retrieve the case where the followers play a set-valued Nash equilibrium problem by setting

$$\begin{cases} P^w_\nu(x_\nu, x_{-\nu}) := \{y_\nu \in X_\nu \mid (y_\nu, x_{-\nu}) \succeq_\nu (x_\nu, x_{-\nu})\},\\ \succeq_\nu \subset X \times X. \end{cases}$$

Then $A^w_\nu(x_{-\nu}) \cap P^w_\nu(x_\nu, x_{-\nu}) = \varnothing$ describes that there is no feasible point $y_\nu \in A^w_\nu(x_{-\nu})$ in which $(y_\nu, x_{-\nu})$ is not preferred over $(x_\nu, x_{-\nu})$. In the language of vector optimization, this says that $(x_\nu, x_{-\nu})$ is *weakly efficient*.

The aim of this paper is to develop an existence theorem for a SLMFG of the form (1.1) where the followers' problem is an abstract economy problem parametrized by the leader's decision. We take the classical approach by showing that the solution map (or the response map) of the lower-level problem has a closed graph. For this, we characterize the solution map with a parametric minimizer set of its associated gap function. We introduce a regularity condition and show that, under the uniform metric, the solution map is upper semicontinuous.

2. Preliminaries

In this section, we recall basic notions in set-valued analysis which include distance between points and sets, convergence of sets, and continuity of set-valued maps. One may

refer to the classical references by Aubin and Frankowska [2], Aliprantis and Border [3], and Rockafellar [4].

Suppose that E and F are subsets of (possibly different) Banach spaces. A set-valued map $T : E \rightrightarrows F$ is said to be

- *closed* if its graph $\mathrm{gr}(T) := \{(x, y) \in E \times F \mid y \in T(x)\}$ is closed in $E \times F$,
- *compact* if $\mathrm{gr}(T)$ is compact in $E \times F$,
- *closed-valued* if $T(x)$ is closed for all $x \in E$,
- *upper semicontinuous* on $D \subset E$ if for every $x \in D$ and each open set $\mathcal{O}$ with $T(x) \subset \mathcal{O}$, there exists a neighborhood $\mathcal{V}$ of x such that $T(x') \subset \mathcal{O}$ for all $x' \in \mathcal{V}$,
- *lower semicontinuous* on $D \subset E$ if for every $x \in D$ and each open set $\mathcal{O}$ with $T(x) \cap \mathcal{O} \neq \varnothing$, there exists a neighborhood $\mathcal{V}$ of x such that $T(x') \cap \mathcal{O} \neq \varnothing$ for all $x' \in \mathcal{V}$,
- *continuous* on $D \subset E$ if it is both upper and lower semicontinuous on D.

The following results (see *e.g.* [2]) regarding $T : E \rightrightarrows F$ are widely known in set-valued analysis:

(i) If T is closed-valued and upper semicontinuous, then it is closed.
(ii) If T is closed and F is compact, then it is upper semicontinuous.
(iii) If F is compact, then T is upper semicontinuous at $x \in E$ if and only if for any sequences (x^n) in E and (y^n) in F with $x^n \to x$ and $y^n \in T(x^n)$ for each $n \in \mathbb{N}$, it holds that $y \in T(x)$.
(iv) If F is compact, then T is lower semicontinuous at $x \in E$ if and only if for any sequence (x^n) in E with $x^n \to x$ and any $y \in T(x)$, there exists $y^n \in T(x^n)$ (for each $n \in \mathbb{N}$) such that $y^n \to y$.

Let us denote with $K(E)$ the family of compact subsets of E. Recall that the distance on E induces a metric on $K(E)$ that preserves the completeness [5]. For any $x \in E$ and $D \in K(E)$, we define the *displacement from* D by

$$d(x, D) := \inf_{y \in D} d(x, y).$$

Observe that $x \in D \iff d(x, D) = 0$. For any $C, D \in K(E)$, the *excess* of C from D is defined by

$$e(C, D) := \sup_{x \in C} \inf_{y \in D} d(x, y).$$

It is important to notice that $e(C, D) \neq e(D, C)$ in general. Finally, the Hausdorff distance between $C, D \in K(E)$ is given by

$$h(C, D) := \max\{e(C, D), e(D, C)\}.$$

For any $D \subset E$ and $\varepsilon > 0$, we adopt the notation

$$\mathcal{U}_\varepsilon[D] := \{x \in E \mid d(x, D) < \varepsilon\}$$

to denote an open ball about a set D. When $C = \{x\}$ is singleton, we prefer to write $\mathcal{U}_\varepsilon[x]$ instead of $\mathcal{U}_\varepsilon[\{x\}]$. One may also adopt its equivalent formulation of the Hausdorff distance [4], which reads

$$h(C, D) = \inf\{\eta > 0 \mid \mathcal{U}_\eta[C] \supset D, \quad \mathcal{U}_\eta[D] \supset C\}. \tag{2.4}$$

It is well-known that $(K(E), h)$ is a metric space which is complete if and only if E is complete [5].

Next, consider a sequence (C^n) of subsets of E. Its *outer* and *inner limits* could be defined, respectively, by

$$\operatorname*{Lim\,sup}_{n \to \infty} C^n := \left\{ x \in E \;\middle|\; \begin{array}{c} x \text{ is an accumulation point of some sequence } (x^n) \\ \text{such that } x^n \in C^n \text{ for all } n \in \mathbb{N}. \end{array} \right\}$$

and

$$\operatorname*{Lim\,inf}_{n\to\infty} C^n := \left\{ x \in E \;\middle|\; \begin{array}{c} x \text{ is the limit of some convergent sequence } (x^n) \\ \text{such that } x^n \in C^n \text{ for all } n \in \mathbb{N}. \end{array} \right\}.$$

We say that $C \subset E$ is the *Painlevé-Kuratowski limit* of (C^n), or symbolically that $C^n \overset{\text{PK}}{\to} C$, if

$$C = \operatorname*{Lim\,sup}_{n\to\infty} C^n = \operatorname*{Lim\,inf}_{n\to\infty} C^n.$$

If C and all C^n's are compact and contained in a bounded set $X \subset E$, then $C^n \overset{\text{PK}}{\to} C$ if and only if $\lim_{n\to\infty} h(C^n, C) = 0$ (see *e.g.*, [4, §4C]). That is, the Painlevé-Kuratowski and the Hausdorff limits coincide for compact sequences over a bounded domain.

The Hausdorff limit could also be used to quantitatively characterize the continuity of a set-valued map $T : E \rightrightarrows F$. That is, if F is compact, then $T : E \rightrightarrows F$ is continuous at $x \in E$ if and only if $\lim_{n\to\infty} h(T(x^n), T(x)) = 0$ for any sequence (x^n) in E with $x^n \to x$. Since this fact is not explicitly presented in any classical textbook, to our best understanding, we include its short proof below for the sake of completeness.

Proposition 2.1. *Let E and F be metric spaces and $T : E \rightrightarrows F$. If F is compact, then T is continuous at $x \in E$ if and only if $\lim_{n\to\infty} h(T(x^n), T(x)) = 0$ for any sequence (x^n) in E with $x^n \to x$.*

Proof. **Part I (only if)**. Assume that T is continuous at $x \in E$ and take any sequence (x^n) converging to x. Fix a positive constant $\eta > 0$. Due to the convergence of (x^n) and the upper semicontinuity of T, we have

$$T(x^n) \subset \mathcal{U}_\eta[T(x)] \tag{2.5}$$

for sufficiently large $n \in \mathbb{N}$.

We next show that for any $\eta > 0$, the inclusion $T(x) \subset \mathcal{U}_\eta[T(x^n)]$ holds for large $n \in \mathbb{N}$. Let us assume to the contrary — that there exists $\eta_0 > 0$ in which there is a subsequence (x^{n_k}) of (x^n) such that

$$T(x) \not\subset \mathcal{U}_\eta[T(x^{n_k})], \qquad (\forall k \in \mathbb{N}).$$

This implies the existence of a sequence (y^k) in $T(x)$ for which

$$d(y^k, T(x^{n_k})) \geq \eta_0, \qquad (\forall k \in \mathbb{N}). \tag{2.6}$$

Due to the compactness of $T(x)$, we may assume without the loss of generality that (y^k) converges to some $y \in T(x)$. Letting $k \to \infty$ in (2.6) causes a contradiction, and we thus have for any $\eta > 0$, the inclusion

$$T(x) \subset \mathcal{U}_\eta[T(x^n)] \tag{2.7}$$

holds for all large $n \in \mathbb{N}$. From (2.5) and (2.7), one obtains $\lim_{n\to\infty} h(T(x^n), T(x)) = 0$.

Part II (if). Suppose that $x \in E$ and $\lim_{n\to\infty} h(T(x^n), T(x)) = 0$ for any sequence (x^n) in E that converges to x. We shall prove the upper and lower semicontinuity separately below.

We begin by showing that T is upper semicontinuous at x. Let us assume to the contrary — that it is not upper semicontinuous. Then there exist an open set $\mathcal{O}' \supset T(x)$, a sequence (x^n) converging to x, and a sequence (y^n) with

$$y^n \in T(x^n) \setminus \mathcal{O}', \qquad (\forall n \in \mathbb{N}). \tag{2.8}$$

If we put $\eta := d(T(x), F \setminus \mathcal{O}') := \inf_{y\in T(x)} d(y, F \setminus \mathcal{O}')$, then the compactness of $T(x)$ ensures that $\eta > 0$. Then we have

$$\mathcal{U}_{\eta/2}[T(x)] \subset \mathcal{O}'.$$

This, together with (2.8), implies that

$$d(y^n, T(x)) \geq \frac{\eta}{2}, \qquad (\forall n \in \mathbb{N}).$$

This contradicts with $\lim_{n\to\infty} h(T(x^n), T(x)) = 0$, and hence T must be upper semicontinuous at x.

Next we show that T is lower semicontinuous at x. Again, we assume to the contrary. Then there exist an open set $\mathcal{O}' \subset F$ with $T(x) \cap \mathcal{O}' \neq \varnothing$ and a sequence (x^n) converging to x such that

$$T(x^n) \cap \mathcal{O}' = \varnothing, (\forall n \in \mathbb{N}).$$

Pick $y \in T(x) \cap \mathcal{O}'$ and $\varepsilon > 0$ in whcih $\mathcal{U}_\varepsilon[y] \subset \mathcal{O}'$. For every $n \in \mathbb{N}$, one has $T(x^n) \cap \mathcal{U}_\varepsilon[x] = \varnothing$ and consequently we get

$$d(y, T(x^n)) \geq \varepsilon.$$

A similar contradiction occurs here, and hence T must be lower semicontinuous at x. □

Finally, our final result relies on the following classical Weierstrass theorem (see *e.g.*, [6, Theorem 2.5]).

Theorem 2.1. *If $D \subset E$ is compact and $\phi : D \to \mathbb{R}$ is lower semicontinuous, then ϕ attains a minimizer.*

3. Stability analysis for abstract economy problems

This section is dedicated to the stability analysis of the lower-level abstract economy. The idea is that a decision w of the leader *selects* a tuple $(A_\nu^w, P_\nu^w)_{\nu\in \mathtt{F}}$ representing an *abstract economy profile*, from some particular class Λ, that defines the followers's abstract economy problem.

3.1. **An existence result.**

For the moment, we neglect the leader's ingredient and focus on the solvability of an abstract economy problem.

Theorem 3.2. *Consider a finite set $\mathtt{F}$ of players. For each $\nu \in \mathtt{F}$, let $\mathcal{X}_\nu$ be a Banach space and $X_\nu \subset \mathcal{X}_\nu$ be nonempty, closed and convex. Suppose, for each $\nu \in \mathtt{F}$, that the following assumptions hold*

(A1) *the maps $A_\nu : X_{-\nu} \rightrightarrows X_\nu$ and $P_\nu : X \rightrightarrows X_\nu$ are upper semicontinuous with closed convex values,*
(A2) *the map A_ν has nonempty values,*
(A3) *the set $E_\nu := \{x \in X \mid A_\nu(x_{-\nu}) \cap P_\nu(x) = \varnothing\}$ is open,*
(A4) *for each $x \in X$, $x_\nu \in A_\nu(x_{-\nu})$ implies $x_\nu \notin P_\nu(x)$.*

Then there is a point $\overline{x} \in X$ such that

$$\overline{x}_\nu \in A_\nu(\overline{x}_{-\nu}), \quad A_\nu(\overline{x}_{-\nu}) \cap P_\nu(\overline{x}) = \varnothing \tag{3.9}$$

for all $\nu \in \mathtt{F}$.

Proof. Let $\nu \in \mathtt{F}$. Define $G_\nu : X \rightrightarrows X_\nu$ by

$$G_\nu(x) := \begin{cases} A_\nu(x_{-\nu}) \cap P_\nu(x) & \text{if } x \in E_\nu, \\ A_\nu(x_{-\nu}) & \text{otherwise.} \end{cases} \tag{3.10}$$

Let $G := \prod_{\nu\in\mathtt{F}} G_\nu$. It is clear that if $x \in X$ satisfies (3.9) for all $\nu \in \mathtt{F}$, then $x \in G(x)$. The converse holds with the use of (A4). This means (3.9) is characterized with $x \in X$ being a fixed point of the map G. From [2, Theorem 1.4.9], G is upper semicontinuous and compact on X with nonempty closed convex values. The existence of such a fixed point then follows from [15, Theorem 1 and Lemma 3]. □

3.2. **A stability result.** In this subsection, we consider a class of *abstract economy profiles*, which are described by tuples $(A_\nu, P_\nu)_{\nu\in\mathtt{F}}$ of constraint-preference maps, where the continuity in the perturbation implies the continuity of the corresponding Nash equilibrium points.

Always consider $\mathtt{F}$ a finite set of players and for each $\nu \in \mathtt{F}$, X_ν is a bounded closed convex subset of a Banach space $\mathcal{X}_\nu$. Consider Λ the class containing $\lambda = (A_\nu, P_\nu)_{\nu\in\mathtt{F}}$, where $A_\nu : X_{-\nu} \rightrightarrows X_\nu$ and $P_\nu : X \rightrightarrows X_\nu$, satisfying the following two conditions

(Λ_1) the assumptions (A1)–(A3) hold,
(Λ_2) there exists $\tau > 0$ such that for all $\nu \in \mathtt{F}$,

$$x_\nu \in A_\nu(x_{-\nu}) \implies \mathcal{U}_\tau[x_\nu] \cap P_\nu(x) = \varnothing.$$

One could directly observe that (Λ_2) implies (A4). Therefore any $\lambda = (A_\nu, P_\nu)_{\nu\in\mathtt{F}} \in \Lambda$ induces a nonempty solution set $\mathsf{NE}(\lambda)$, where

$$\mathsf{NE}(\lambda) := \{x \in X \mid x_\nu \in A_\nu(x_{-\nu}), \quad A_\nu(x_{-\nu}) \cap P_\nu(x) = \varnothing, \quad \forall \nu \in \mathtt{F}\}.$$

Given $\mathcal{E} := (\mathcal{E}_\nu)_{\nu\in\mathtt{F}}$, where for each $\nu \in \mathtt{F}$, $\mathcal{E}_\nu \subset X$ is open relative to X, we define a subclass $\Lambda(\mathcal{E}) \subset \Lambda$ by

$$\Lambda(\mathcal{E}) := \{\lambda \in \Lambda \mid E_\nu(\lambda) = \mathcal{E}_\nu, \quad \forall \nu \in \mathtt{F}\},$$

where $E_\nu(\lambda) := \{x \in X \mid A_\nu(x_{-\nu}) \cap P_\nu(x) = \varnothing\}$. On this subset $\Lambda(\mathcal{E})$, we impose a distance function $\rho : \Lambda(\mathcal{E}) \times \Lambda(\mathcal{E}) \to \mathbb{R}_+$ given by

$$\rho(\lambda, \hat{\lambda}) = \sum_{\nu\in\mathtt{F}} \left[\sup_{x\in X} h(A_\nu(x_{-\nu}), \hat{A}_\nu(x_{-\nu})) + \sup_{x\in X} h(P_\nu(x), \hat{P}_\nu(x))\right], \tag{3.11}$$

where $\lambda = (A_\nu, P_\nu)_{\nu\in\mathtt{F}}$ and $\hat{\lambda} = (\hat{A}_\nu, \hat{P}_\nu)_{\nu\in\mathtt{F}}$. The pair $(\Lambda(\mathcal{E}), \rho)$ is clearly a metric space, but its completeness is not so easy to see. In fact, to prove its completeness, we need to assume a regularity condition below.

Definition 3.1. *A closed subset $\Psi \subset \Lambda(\mathcal{E})$ is said to be* regular w.r.t. $\mathcal{E}$ *if*

(Ψ_1) *the condition* (Λ_2) *holds uniformly,* i.e. *with the same $\tau > 0$ for any $\lambda \in \Psi$,*
(Ψ_2) *there exists $\alpha > 0$ such that for any $\varepsilon > 0$, any $\lambda = (A_\nu, P_\nu)_{\nu\in\mathtt{F}} \in \Psi$, and any $x \in E_\nu = \mathcal{E}_\nu$, it holds*

$$\mathcal{U}_\varepsilon[A_\nu(x_{-\nu})] \cap \mathcal{U}_\varepsilon[P_\nu(x)] \subset \mathcal{U}_{\alpha\varepsilon}[A_\nu(x_{-\nu}) \cap P_\nu(x)].$$

Following directly from the definition, one could observe that the above regularity always holds in finite dimensional spaces when the intersections $A_\nu(x_{-\nu}) \cap P_\nu(x)$ are bounded.

Lemma 3.1. *If $\Psi \subset \Lambda(\mathcal{E})$ is regular w.r.t. $\mathcal{E}$, then (Ψ, ρ) is complete.*

Proof. Let $(\lambda^n) = (A^n_\nu, P^n_\nu)^{n\in\mathbb{N}}_{\nu\in\mathtt{F}}$ be a Cauchy sequence in Ψ. For every $x \in X$ and $\nu \in \mathtt{F}$, the sequence $(A^n_\nu(x_{-\nu}))^{n\in\mathbb{N}}$ of compact sets is Cauchy w.r.t. to the Hausdorff metric on $K(X_\nu)$, which in turn converges, in the Hausdorff metric, to a compact set $\hat{A}_\nu(x_{-\nu}) \in K(X_\nu)$. In the same way, the sequence $(P^n_\nu(x))^{n\in\mathbb{N}}$ converges to a set $\hat{P}_\nu(x) \in K(X_\nu)$ in the Hausdorff metric. Through these limits, we have defined the maps $\hat{A}_\nu : X_{-\nu} \rightrightarrows X_\nu$

and $\hat{P}_\nu : X \rightrightarrows X_\nu$ which produce compact convex values. The goal is to prove that $\hat{\lambda} := (\hat{A}_\nu, \hat{P}_\nu)_{\nu\in\mathtt{F}}$ belongs to Λ, through the three claims below.

Claim 1: The maps $\hat{A}_\nu$ and $\hat{P}_\nu$ are upper semicontinuous.

Take any sequences (x^n) and (y^n) in X that are convergent, respectively, to $\hat{x} \in X$ and $\hat{y} \in X$ with the property that $y^n \in \hat{A}_\nu(x^n)$ for all $n \in \mathbb{N}$. It suffices to show that $\hat{y} \in \hat{A}_\nu(\hat{x})$. Now, for any $\nu \in \mathtt{F}$ and $r, n \in \mathbb{N}$, we get

$$d(y^n, \hat{A}_\nu(\hat{x})) \le d(y^n, \hat{A}_\nu(x^n)) + h(\hat{A}_\nu(x^n), A^r_\nu(x^n)) + e(A^r_\nu(x^n), \hat{A}_\nu(\hat{\nu})) \tag{3.12}$$

$$= h(\hat{A}_\nu(x^n), A^r_\nu(x^n)) + e(A^r_\nu(x^n), \hat{A}_\nu(\hat{x})). \tag{3.13}$$

Let $\varepsilon > 0$ be arbitrary. Since $\lim_{r\to\infty} h(A^r_\nu(x), \hat{A}_\nu(x))$ uniformly on $x \in X$, there exists $R > 0$ in which

$$h(\hat{A}_\nu(x), A^r_\nu(x)) < \frac{\varepsilon}{2} \tag{3.14}$$

for any $r > R$ and $x \in X$. Take any $r > R$, the above inequality gives

$$A^r_\nu(\hat{x}) \subset \mathcal{U}_{\varepsilon/\alpha}[\hat{A}_\nu(\hat{x})].$$

Together with the facts that each map A^r_ν is upper semicontinuous and $x^n \to \hat{x}$, there exists $N > 0$ in which

$$A^r_\nu(x^n) \subset \mathcal{U}_{\varepsilon/2}[\hat{A}_\nu(\hat{x})]$$

for all $n > N$. This means, for any $r, n > \max\{R, N\}$, it holds that

$$e(A^r_\nu(x^n), \hat{A}_\nu(\hat{x})) < \frac{\varepsilon}{2}. \tag{3.15}$$

Combining (3.12)–(3.13) with (3.14) and (3.15), we obtain that

$$d(y^n, \hat{A}_\nu(\hat{x})) < \varepsilon$$

for each $n > \max\{R, N\}$. It implies that $\hat{y} \in \hat{A}_\nu(\hat{x})$, and we conclude that $\hat{A}_\nu$ is upper semicontinuous.

With a similar technique, we could show that $\hat{P}_\nu$ is upper semicontinuous, so that Claim 1 is proved.

Claim 2: $E_\nu(\hat{\lambda}) = \mathcal{E}_\nu$ for all $\nu \in \mathtt{F}$.

This claim follows directly from the regularity of Ψ w.r.t $\mathcal{E}$.

Claim 3: For $x \in X$, $x_\nu \in \hat{A}_\nu(x_{-\nu})$ implies $\mathcal{U}_\tau[x_\nu] \cap \hat{P}_\nu(x) = \varnothing$ for all $x \in X$.

Let $\nu \in \mathtt{F}$ and take any $\hat{x} \in X$ with $\hat{x}_\nu \in \hat{A}_\nu(\hat{x}_{-\nu})$. Due to the equivalence from Proposition 2.1, using the PK convergence of $(A^n_\nu(\hat{x}_{-\nu}))$ to its limit $\hat{A}_\nu(\hat{x}_{-\nu})$, there exists a sequence (x^n) in X such that $x^n \to \hat{x}$ and $x^n_\nu \in A^n_\nu(\hat{x}_{-\nu})$ for all $n \in \mathbb{N}$. Applying (Ψ_1) to the point $(x^n_\nu, \hat{x}_{-\nu})$ with $A^n_\nu(\hat{x}_{-\nu})$, we obtain

$$\tau \le d(x^n_\nu, P^n_\nu(\hat{x})) \le d(x^n_\nu, \hat{P}_\nu(\hat{x})) + h(P^n_\nu(\hat{x}), P_\nu(\hat{x}))$$

for large $n \in \mathbb{N}$. Letting $n \to \infty$, we have $\tau \le d(\hat{x}_\nu, \hat{P}_\nu(\hat{x}))$, which is equivalent to saying $\mathcal{U}_\tau[\hat{x}_\nu] \cap P_\nu(\hat{x}) = \varnothing$. The claim is thus proved.

Accumulating the Claims 1–3, one conclude that (Λ_1) and (Λ_2) are satisfied at the limit $\hat{\lambda}$. This proves $\hat{\lambda} \in \Lambda$ and the closedness of Ψ guarantees that it is complete. □

Following the proof of Theorem 3.2, one have a fixed point characterization of elements in $\mathsf{NE}(\cdot)$. We define a function $\mathcal{R} : \Lambda \times X \to \mathbb{R}_+$ by

$$\mathcal{R}(\lambda; x) := \max_{\nu \in \mathtt{F}} d(x_\nu, G_\nu(\lambda; x)),$$

$$\text{where} \quad G_\nu(\lambda; x) := \begin{cases} A_\nu(x_{-\nu}) \cap P_\nu(x) & \text{if } x \in E_\nu(\lambda), \\ A_\nu(x_{-\nu}) & \text{otherwise,} \end{cases}$$

for any $\lambda = (A_\nu, P_\nu)_{\nu \in \mathtt{F}} \in \Lambda$. One may observe that $\mathcal{R}(\lambda; \cdot)$ is a gap function for the game λ, *i.e.* it is nonnegative and $\mathcal{R}(\lambda; x) = 0$ if and only if $x \in \mathsf{NE}(\lambda)$.

Lemma 3.2. *The gap function $\mathcal{R} : \Lambda \times X \to \mathbb{R}_+$ is lower semicontinuous on $\Psi \times X$, provided that $\Psi \subset \Lambda(\mathcal{E})$ is regular w.r.t. $\mathcal{E}$.*

Proof. Let (λ^n), $\lambda^n = (A^n_\nu, P^n_\nu)_{\nu \in \mathtt{F}}$, be a sequence in Ψ that is convergent to $\hat{\lambda} = (\hat{A}_\nu, \hat{P}_\nu) \in \Psi$ w.r.t. ρ.

Claim: For any $\nu \in \mathtt{F}$, $G_\nu(\lambda^n; x) \to G_\nu(\hat{\lambda}; x)$ uniformly over $x \in X$.

It is sufficient to show that $A^n_\nu(x_{-\nu}) \cap P^n_\nu(x) \to \hat{A}_\nu(x_{-\nu}) \cap \hat{P}_\nu(x)$ uniformly on $x \in E_\nu$. It is clear that we have

$$A^n_\nu(x) \to \hat{A}_\nu(x) \quad \text{and} \quad P^n_\nu(x) \to \hat{P}_\nu(x) \tag{3.16}$$

uniformly on $x \in X$. Let us fix $\nu \in \mathtt{F}$ and $x \in E_\nu$. For any $\varepsilon > 0$, and $n \in \mathbb{N}$ sufficiently large, we have

$$\begin{aligned} &\hat{A}_\nu(x_{-\nu}) \subset \mathcal{U}_{\varepsilon/\alpha}[A^n_\nu(x_{-\nu})], \quad A^n_\nu(x_{-\nu}) \subset \mathcal{U}_{\varepsilon/\alpha}[\hat{A}_\nu(x_{-\nu})] \\ &\hat{P}_\nu(x) \subset \mathcal{U}_{\varepsilon/\alpha}[P^n_\nu(x)], \qquad\quad P^n_\nu(x) \subset \mathcal{U}_{\varepsilon/\alpha}[\hat{P}_\nu(x)]. \end{aligned}$$

For large $n \in \mathbb{N}$, the regularity of Ψ yields

$$\begin{aligned} \hat{A}_\nu(x_{-\nu}) \cap \hat{P}_\nu(x) &\subset \mathcal{U}_{\varepsilon/\alpha}[A^n_\nu(x_{-\nu})] \cap \mathcal{U}_{\varepsilon/\alpha}[P^n_\nu(x)] \subset \mathcal{U}_\varepsilon[A^n_\nu(x_{-\nu}) \cap P^n_\nu(x)] \\ A^n_\nu(x_{-\nu}) \cap P^n_\nu(x) &\subset \mathcal{U}_{\varepsilon/\alpha}[\hat{A}_\nu(x_{-\nu})] \cap \mathcal{U}_{\varepsilon/\alpha}[\hat{P}_\nu(x)] \ \subset \mathcal{U}_\varepsilon[\hat{A}_\nu(x_{-\nu}) \cap \hat{P}_\nu(x)]. \end{aligned}$$

This means we have

$$h(\hat{A}_\nu(x_{-\nu}) \cap \hat{P}_\nu(x), A^n_\nu(x_{-\nu}) \cap P^n_\nu(x)) < \varepsilon$$

whenever $n \in \mathbb{N}$ is sufficiently large. Since this is uniform over $x \in E_\nu$, we conclude that

$$\sup_{x \in E_\nu} h(\hat{A}_\nu(x_{-\nu}) \cap \hat{P}_\nu(x), A^n_\nu(x_{-\nu}) \cap P^n_\nu(x)) \to 0.$$

Together with the uniform convergence of $(A^n_\nu(x))$ over $x \in X \setminus E_\nu$, which could be observed in (3.16). This proves the claim.

Now, note that $G_\nu(\lambda^n; \cdot)$ and $G_\nu(\hat{\lambda}; \cdot)$ are upper semicontinuous, and hence the functions $x \mapsto d(x, G_\nu(\lambda^n; x))$ and $x \mapsto d(x, G_\nu(\hat{\lambda}; \cdot))$ are both lower semicontinuous (by [2, Corollary 1.4.17]). Take any convergent sequence (x^n) in X with the limit $\hat{x} \in X$. Let $\varepsilon > 0$. For $n \in \mathbb{N}$ sufficiently large, we have

$$d(x^n, G_\nu(\hat{\lambda}; x^n)) > d(\hat{x}, G_\nu(\hat{\lambda}; \hat{x})) - \frac{\varepsilon}{2}.$$

Using the compactness, for any $n \in \mathbb{N}$, there exists $u^n \in G_\nu(\lambda^n; x^n)$ such that

$$d(x^n, u^n) = d(x^n, G_\nu(\lambda^n; x^n)).$$

From the convergence proved in Claim 1, for large enough $n \in \mathbb{N}$, we could pick $v^n \in G_\nu(\hat{\lambda}; x^n)$ such that

$$d(u^n, v^n) = d(u^n, G_\nu(\hat{\lambda}; x^n)) \leq h(G_\nu(\lambda^n; x^n), G_\nu(\hat{\lambda}; x^n)) < \frac{\varepsilon}{2}.$$

Therefore, we have

$$\begin{aligned} d(x^n, G_\nu(\lambda^n; x^n)) &= d(x^n, u^n) \\ &\geq d(x^n, v^n) - d(v^n, u^n) \\ &\geq d(x^n, G_\nu(\hat{\lambda}; x^n)) - d(v^n, u^n) \\ &> d(\hat{x}, G_\nu(\hat{\lambda}; \hat{x})) - \varepsilon \end{aligned}$$

for large $n \in \mathbb{N}$. Repeat the process for all $\nu \in \mathtt{F}$, we get

$$\mathcal{R}(\lambda^n; x^n) > \mathcal{R}(\hat{\lambda}; \hat{x}) - \varepsilon$$

for large $n \in \mathbb{N}$. Thus, the function $\mathcal{R}$ is lower semicontinuous on $\Psi \times X$. □

We finally have all the tools to prove our main stability theorem. We now state and prove the continuity of $\mathsf{NE}(\cdot)$ based on Lemmas 3.1 and 3.2.

Theorem 3.3. *The map $\lambda \mapsto \mathsf{NE}(\lambda)$ is upper semicontinuous on $\Psi \subset \Lambda(\mathcal{E})$, provided that Ψ is regular w.r.t. $\mathcal{E}$.*

Proof. Observe that, for each $\lambda = (A_\nu, P_\nu)_{\nu \in \mathtt{F}} \in \Psi$, one may now deduce that

$$\mathsf{NE}(\lambda) = \mathsf{S}(\lambda) := \{x \in X \mid \mathcal{R}(\lambda; x) = 0\} = \operatorname*{arg\,min}_{x \in X} \mathcal{R}(\lambda; x).$$

This set above is nonempty for all $\lambda \in \Psi$ in view of Theorem 3.2. To show that $\mathsf{NE} = \mathsf{S}$ is upper semicontinuous, we take a convergent sequence (λ^n) in Ψ with the limit $\hat{\lambda} \in \Psi$. Let (x^n) be a sequence in X such that $x^n \in \mathsf{S}(\lambda^n)$, for all $n \in \mathbb{N}$, and is convergent to $\hat{x} \in X$. This means $\mathcal{R}(\lambda^n; x^n) = 0$ for all $n \in \mathbb{N}$. The lower semicontinuity of $\mathcal{R}$ implies that $\mathcal{R}(\hat{\lambda}; \hat{x}) = 0$ and that $\hat{x} \in \mathsf{S}(\hat{\lambda})$. Therefore, the map $\mathsf{NE} = \mathsf{S}$ is upper semicontinuous on Ψ. □

3.3. **An example.** We finish this section with an example to illustrate Theorem 3.3.

Example 3.1. *Here, we consider a game of two players $\nu \in \mathtt{F} = \{1, 2\}$. For each $\nu \in \mathtt{F}$, put $\mathcal{X}_\nu = \mathbb{R}$, and $X_\nu = [0, 1]$. This gives $X = [0, 1] \times [0, 1]$. We let*

$$\mathcal{E}_1 = \{(x_1, x_2) \in X \mid x_1 > 0.5\} \quad \textit{and} \quad \mathcal{E}_2 = \{(x_1, x_2) \in X \mid x_2 < 0.2\},$$

and set $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$. Next we define the games $\lambda^n = (A^n_\nu, P^n_\nu)_{\nu=1,2}$. For $n = 0$, let

$$A^0_1(x_2) = [0.9, 0.9 + 0.1x_2], \qquad P^0_1(x_1, x_2) = \begin{cases} [0.6, 1] & \text{if } x_1 \leq 0.5 \\ \varnothing & \text{if } x_1 > 0.5 \end{cases}$$

$$A^0_2(x_1) = [0.1x_1, 0.1], \qquad P^0_2(x_1, x_2) = \begin{cases} [0, 0.1] & \text{if } x_2 \geq 0.2 \\ \varnothing & \text{if } x_2 < 0.2 \end{cases}$$

and, for $n = 1, 2, \ldots$, define

$$A^n_1(x_2) = [0.9 - \tfrac{0.05}{n}, 0.9 + 0.1x_2], \quad P^n_1(x_1, x_2) = \begin{cases} [0.6 + \frac{0.05}{n}, 1] & \text{if } x_1 \leq 0.5 \\ \varnothing & \text{if } x_1 > 0.5 \end{cases}$$

$$A^n_2(x_1) = [0.1x_1, 0.1 + \tfrac{0.05}{n}], \qquad P^n_2(x_1, x_2) = \begin{cases} [0, 0.1 + \frac{0.05}{n}] & \text{if } x_2 \geq 0.2 \\ \varnothing & \text{if } x_2 < 0.2 \end{cases}$$

Moreover, we set

$$\Psi = \{\lambda^n \mid n = 0, 1, 2, \ldots\}.$$

For any player $\nu \in \mathtt{F}$ and any $n = 0, 1, 2, \ldots$, it is not difficult to see that

$$E_\nu(\lambda^n) = \{x \in X \mid A_\nu^n(x_{-\nu}) \cap P_\nu^n(x) = \varnothing\} = \mathcal{E}_\nu. \tag{3.17}$$

We next prove, in the followin claims, that λ^n satisfies (Λ_1), and that the collection Ψ satisfies (Ψ_1) uniformly in the sense of (Ψ_1).

Claim A: *λ^n satisfies (Λ_1) for all $n = 0, 1, 2, \ldots$.*

Proof of Claim A. *We need to show that* (A1)–(A3) *hold for any $n = 0, 1, 2, \ldots$. It is clear that both A_ν^n and P_ν^n have closed convex values. Since X_ν is compact and the graphs of both maps are closed, they are upper semicontinuous. Thus,* (A1) *holds. The map A_ν^n clearly has nonempty values, implying* (A2)*. Moreover, the equation* (3.17) *ensures* (A3) *as $E_\nu(\lambda^n) = \mathcal{E}_\nu$ is open in X_ν. Thus, the Claim A is proved.*

Claim B: *There is $\tau > 0$ such that for any $n = 0, 1, 2, \ldots$, the condition (Λ_2) holds.*

Proof of Claim B. *Take an arbitrary $n = 0, 1, 2, \ldots$ and any $\nu = 1, 2$. For each feasible action $x_\nu \in A_\nu(x_{-\nu})$, we have $P_\nu(x_1, x_2) = \varnothing$. This means (Ψ_2) holds for any $\tau > 0$, and hence the Claim B is proved.*

With Claims A and B, we have actually showed that $\Psi \subset \Lambda(\mathcal{E}) \subset \Lambda$ and that Ψ enjoys the property (Ψ_1). The condition (Ψ_2) is also verified since X_ν is of a finite dimension. Therefore the collection Ψ is regular w.r.t. $\mathcal{E}$, and Lemma 3.1 implies (Ψ, ρ) is a complete metric space with ρ defined as in (3.11)*. Under such a metric, we have*

$$\begin{aligned}\rho(\lambda^n, \lambda^0) &= \sup_{x_2} h(A_1^n(x_2), A_1^0(x_2)) + \sup_x h(P_1^n(x), P_1^0(x)) \\ &\quad + \sup_{x_1} h(A_2^n(x_1), A_2^0(x_1)) + \sup_x h(P_2^n(x), P_2^0(x)) \\ &= \frac{0.05}{n} + \frac{0.05}{n} + \frac{0.05}{n} + \frac{0.05}{n} = \frac{0.2}{n}.\end{aligned}$$

As $n \to \infty$, we obtain $\rho(\lambda^n, \lambda^0) \to 0$.

Next, we evaluate the sets of Nash equilibrium points of the games λ^n's. At $n = 0$, we get

$$\mathsf{NE}(\lambda^0) = \{(x_1, x_2) \in X \mid 0.9 \le x_1 \le 0.9 + 0.1x_2, \quad 0.1x_1 \le x_2 \le 0.1\}$$

and at $n = 1, 2, \ldots$, we get

$$\mathsf{NE}(\lambda^n) = \{(x_1, x_2) \in X \mid 0.9 - \tfrac{0.05}{n} \le x_1 \le 0.9 + 0.1x_2, \quad 0.1x_1 \le x_2 \le 0.1 + \tfrac{0.05}{n}\}.$$

Note that $\mathsf{NE}(\lambda^n) \supset \mathsf{NE}(\lambda^0)$ for all $n = 1, 2, \ldots$, and these solution sets are illustrated in the Figure 3.1 below.

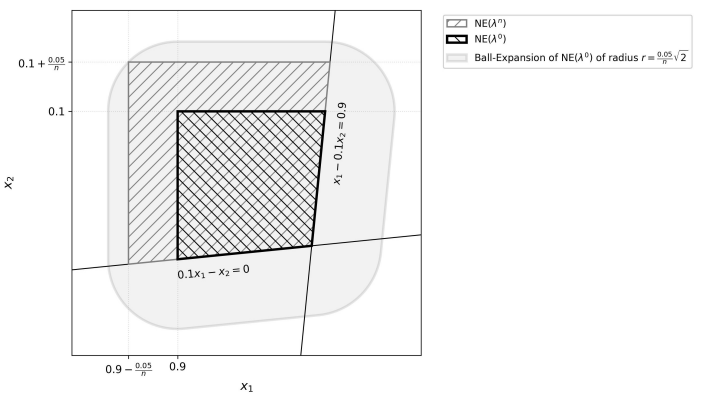


FIGURE 3.1. The sets $\mathsf{NE}(\lambda^0)$, $\mathsf{NE}(\lambda^n)$ and the ball expansion about $\mathsf{NE}(\lambda^0)$ or radius $r = \frac{0.05}{n}\sqrt{2}$.

Based on (2.4)*, we obtains that*

$$h(\mathsf{NE}(\lambda^n), \mathsf{NE}(\lambda^0)) = \frac{0.05}{n}\sqrt{2},$$

which converges to 0 *as* $n \to \infty$*. One could also consult Figure 3.1 for the above calculation. Due to the construction of* Ψ*, combining the limits* $\lambda^n \xrightarrow{\rho} \lambda^0$ *and* $\mathsf{NE}(\lambda^n) \xrightarrow{h} \mathsf{NE}(\lambda^0)$ *with the Proposition 2.1, we arrive at a conclusion that the solution map*

$$\lambda \in \Psi \mapsto \mathsf{NE}(\lambda)$$

is continuous. This confirms our finding in Theorem 3.3.

4. An existence result for SLMFG with direct preference maps

This final section presents an existence result for the SLMFG with direct preference maps of the form (1.1), with a more specific structure of the followers' problem. The theorem is, in fact, just an application of the Weierstraß theorem combined with the stability result from Theorem 3.3.

Theorem 4.4. *Suppose that*

(a) W *is a topological space and* $Y \subset W$ *is closed,*
(b) $F : W \times X \to \mathbb{R}$ *is lower semicontinuous,*
(c) $\sigma : W \to \Psi$ *is continuous, where* $\Psi \subset \Lambda(\mathcal{E})$ *is regular w.r.t.* $\mathcal{E}$*.*

Then the following SLMFG has a solution:

$$\begin{cases} \min\limits_{w,x} & F(w,x) & \text{(4.18a)} \\ s.t. & w \in Y & \text{(4.18b)} \\ & \sigma(w) = (A^w_\nu, P^w_\nu)_{\nu\in\mathtt{F}} & \text{(4.18c)} \\ & \forall \nu \in \mathtt{F}: & \text{(4.18d)} \\ & \lfloor\, x_\nu \in A^w_\nu(x_{-\nu}), \quad A^w_\nu(x_{-\nu}) \cap P^w_\nu(x) = \varnothing. & \text{(4.18e)} \end{cases}$$

Proof. One could observe that the constraints (4.18c)–(4.18e) reduce simply to $x \in \mathsf{NE}(\sigma(w))$, or equivalently, to $(w,x) \in \mathrm{gr}(\mathsf{NE} \circ \sigma)$. This means the SLMFG (4.18) reduces to

$$\begin{cases} \min\limits_{w,x} & F(w,x) & \text{(4.19a)} \\ \text{s.t.} & w \in Y & \text{(4.19b)} \\ & (w,x) \in \mathrm{gr}(\mathsf{NE} \circ \sigma). & \text{(4.19c)} \end{cases}$$

From Theorem 3.3, the response map NE is upper semicontinuous. Since σ is continuous, the composition $\mathsf{NE} \circ \sigma : W \rightrightarrows X$ is again upper semicontinuous. As a consequence, the graph of $\mathsf{NE} \circ \sigma$ is closed due to the compactness of X. The Theorem 2.1 then guarantees that (4.19), and hence (4.18), has a solution. □

In the formulation of SLMFG (4.18), the leader's action w is passed to the followers through the *signal map* σ, which explicitly selects an abstract economy profile $\lambda \in \Psi$. One could also present Theorem 4.4 (and the problem (4.18) herein) more transparently by adopting the parametric approach, *i.e.*, each follower $\nu \in \mathtt{F}$ has the parametrized constraint map $A_\nu : W \times X_{-\nu} \rightrightarrows X_\nu$ and the parametrized preference map $P_\nu : W \times X \rightrightarrows X_\nu$. An alternative presentation of Theorem 4.4 now reads: If the conditions (a) and (b) hold, and the tuple $(A_\nu(w,\cdot), P_\nu(w,\cdot))_{\nu\in\mathtt{F}}$ belongs to Ψ, then the corresponding SLMFG of the form (1.1) has a solution.

Conclusion

In this paper, we have established a general existence and stability results for an abstract economy as the tools to establish an existence criteria for a more complicated game, namely the single-leader multi-follower game. By embedding the followers' abstract economy profiles in an appropriate metric space and demonstrating upper semicontinuity of the corresponding response map, we circumvent the technical limitations of traditional GNEP-based approaches. Under mild regularity and continuity assumptions, the leader's optimization problem admits a solution via a Weierstrass argument. Our results unify and extend several strands of the literature on existence theorems in bilevel and multi-agent equilibrium problems.

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