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Double Inertial Splitting Algorithm with Correction Terms for Split Variational Inclusion Problems and Image Restoration

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ABSTRACT. We develop a forward-backward splitting method combining double inertial extrapolation with a correction step to address the split variational inclusion problem (SVIP) in real Hilbert spaces. A parameter $\delta \in [0, 1)$ governs the correction step, while a self-adaptive rule for λ_n eliminates the need to bound $\|A\|$ prior to execution. Weak convergence of the sequence is established under mild conditions. The framework specializes naturally to the split feasibility problem, yielding a convergent algorithm. Computational tests on motion-blurred medical image reconstruction demonstrate the practical advantage: over 500 iterations, the proposed method achieves superior PSNR, SSIM, and MSE relative to several recently proposed inertial splitting algorithms.

1. INTRODUCTION

Let H_1 and H_2 be real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and induced norms $\|\cdot\|$, and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone operators, and let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be μ-inverse strongly monotone and ν-inverse strongly monotone mappings, respectively. The **Split Monotone Inclusion Problem (SMIP)**, introduced by Moudafi [28], is to find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*), \tag{1.1}$$

and such that $y^* = Ax^* \in H_2$ satisfies

$$0 \in f_2(y^*) + B_2(y^*). \tag{1.2}$$

We denote the solution set of (1.1)–(1.2) by

$$\Omega := \big\{x^* \in H_1 : 0 \in f_1(x^*) + B_1(x^*),\ Ax^* \in H_2,\ 0 \in f_2(Ax^*) + B_2(Ax^*)\big\},$$

and we assume throughout that $\Omega \neq \emptyset$. The SMIP serves as a unifying framework encompassing several important special cases—among them the split variational inequality problem, the split zero problem, the split minimization problem, and the split feasibility problem [1, 4, 5, 8, 14, 17, 21, 24]. Problems of this type arise prominently in intensity-modulated radiation therapy (IMRT) planning [12, 13], as well as in signal processing, sensor networks, computerized tomography, and data compression [9, 19].

When $f_1 \equiv 0$ and $f_2 \equiv 0$, the SMIP (1.1)–(1.2) reduces to the **Split Variational Inclusion Problem (SVIP)**: find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.3}$$

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and

$$y^* = Ax^* \in H_2 \quad \text{with} \quad 0 \in B_2(Ax^*). \tag{1.4}$$

The foundational iterative method for the SVIP is due to Byrne et al. [11], whose algorithm generates a sequence $\{x_n\}$ via

$$x_{n+1} = J_\lambda^{B_1}\left(x_n + \gamma A^*\left(J_\lambda^{B_2} - I\right)Ax_n\right), \quad \lambda > 0, \quad \gamma \in \left(0, \frac{2}{\|A^*A\|}\right), \tag{1.5}$$

where $J_\lambda^B = (I + \lambda B)^{-1}$ denotes the resolvent of B with parameter $\lambda > 0$, and established weak convergence under standard assumptions. For the general SMIP, Moudafi [28] proposed an analogous forward-backward-type iteration, obtaining weak convergence in real Hilbert spaces.A key drawback common to these foundational schemes is that γ in (1.5) must satisfy an explicit bound involving $\|A\|$, or equivalently the spectral radius of A^*A. In large-scale settings, computing or even reliably estimating this quantity is often prohibitively expensive [39, 33].

To overcome the norm-dependency drawback, **self-adaptive step-size rules** have become a central research direction. López et al.[26] introduced a relaxed CQ algorithm for the split feasibility problem with self adaptive step size. Tang and Gibali [33] extended this idea to the SVIP by defining auxiliary functions

$$F(x) = A^*(I - J_r^{B_2})Ax, \qquad H(x) = (I - J_r^{B_1})x, \qquad r > 0, \tag{1.6}$$

and introduced the following algorithm:

$$y_n = (1 - \beta_n)x_n + \beta_n J_r^{B_1} x_n, \tag{1.7}$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\left(I - \tau_n A^*(I - J_r^{B_2})A\right) y_n, \tag{1.8}$$

where

$$\beta_n = \frac{\rho_n\, h(x_n)}{\|F(x_n)\|^2 + \|H(x_n)\|^2}, \qquad \tau_n = \frac{\rho_n\, f(y_n)}{\|F(y_n)\|^2 + \|H(y_n)\|^2},$$

with $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ and $h(x) = \frac{1}{2}\|(I - J_r^{B_1})x\|^2$, chosen self-adaptively without requiring $\|A\|$. Their algorithm achieves weak convergence.

Running parallel to these developments, inertial acceleration tracing its origins to Polyak's heavy-ball method and later formalized in the proximal setting by Alvarez and Attouch [3] has been embedded into a wide class of splitting schemes. The guiding principle is to bias the current search direction using the displacement $x_n - x_{n-1}$ from the preceding step.

Yao et al. [39] applied this strategy to the SMIP, proposing an algorithm that begins each iteration by forming the extrapolated point

$$w_n := x_n + \theta_n(x_n - x_{n-1}), \tag{1.9}$$

where $\theta_n \in [0, \bar{\theta}_n]$ is a carefully controlled inertial factor satisfying

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta,\ \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|^2}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{if } x_n = x_{n-1}, \end{cases}$$

with $\{\varepsilon_n\} \subset \ell^1$ and $\theta \in [0, 1)$. The update step then reads

$$x_{n+1} = J_\lambda^{B_1}(w_n + \gamma_n A^*(T - I)Aw_n), \tag{1.10}$$

where $T := J_\lambda^{B_2}(I - \lambda f_2)$, $U := J_\lambda^{B_1}(I - \lambda f_1)$, and the step size

$$\gamma_n := \begin{cases} \dfrac{\rho_n \|(T-I)Aw_n\|^2}{\|A^*(T-I)Aw_n\|^2}, & \text{if } (T-I)Aw_n \neq 0, \\ \gamma, & \text{otherwise,} \end{cases} \tag{1.11}$$

is fully self-adaptive and independent of $\|A\|$. Under mild assumptions, specifically $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 1$ and $0 < \lambda < 2\alpha$, weak convergence of $\{x_n\}$ to a point in Ω is established in [39], and numerical experiments confirm that an appropriately chosen inertial factor θ_n significantly accelerates convergence compared to the non-inertial case $\theta_n = 0$.

More recently, **double inertial extrapolation** has emerged as a further acceleration strategy. Wang et al. [37] introduced a modified Tseng splitting method for the monotone inclusion problem $0 \in (A + B)x^*$, incorporating two distinct momentum sequences $\{\alpha_n\}$ and $\{\beta_n\}$. In their algorithm, two extrapolated points are formed simultaneously:

$$\begin{aligned} w_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ z_n &= x_n + \beta_n(x_n - x_{n-1}), \\ y_n &= (I + \lambda_n B)^{-1}(I - \lambda_n A)w_n, \\ x_{n+1} &= (1-\theta_n)z_n + \theta_n(y_n - \lambda_n(Ay_n - Aw_n)), \end{aligned} \tag{1.12}$$

where the step size satisfies

$$\lambda_{n+1} = \begin{cases} \min\left\{ \dfrac{(\mu_n + \mu)\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\ \lambda_n + p_n \right\}, & \text{if } Aw_n \neq Ay_n, \\ \lambda_n + p_n, & \text{otherwise,} \end{cases} \tag{1.13}$$

with $\sum_{n=1}^{\infty} p_n < \infty$ and $\lim_{n\to\infty} \mu_n = 0$. This framework simultaneously achieves weak convergence, a nonasymptotic $O(1/\sqrt{n})$ convergence rate, strong convergence under strong monotonicity, and Q-linear convergence rate [37]. Crucially, the inertial factor α_n in (1.12)–(1.12) is allowed to equal 1, which is not permitted in single-inertial methods [37].

Drawing on the above developments, we construct an algorithm for the SVIP (1.3)–(1.4) that consolidates self-adaptive step sizing, double inertial extrapolation, and a correction mechanism within a single iterative framework.

Specifically, we introduce a splitting algorithm with double inertial and correction terms, in which the extrapolated point is defined by

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) + \delta(1 + \alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}), \tag{1.14}$$

combining a double inertial term with a correction term controlled by $\delta \in [0, 1)$. The forward-backward step and correction update then read

$$y_n = J_{\beta_n}^{B_1}\Big(w_n - \lambda_n A^*(I - J_{\beta_n}^{B_2})Aw_n\Big), \tag{1.15}$$

$$x_{n+1} = y_n - \lambda_n\Big(A^*(I - J_{\beta_n}^{B_2})Ay_n - A^*(I - J_{\beta_n}^{B_2})Aw_n\Big), \tag{1.16}$$

where the step size λ_n is updated self-adaptively, requiring no prior knowledge of the operator norm $\|A\|$. The proposed method inherits the practical advantage of norm-independence, while the double inertial mechanism and correction term yield substantially faster convergence in practice. We establish weak convergence theorems in real Hilbert spaces under mild assumptions on the algorithm parameters, and provide numerical experiments on image recovery via split feasibility problem to validate the theoretical results.

The remainder of the paper is organized as follows. Section 2 collects preliminary definitions and key lemmas used throughout the analysis. Section 3 presents the proposed algorithm and establishes its convergence properties. Section 4 reports applications on image recovery via split feasibility problem, and concluding are given in Section 5.

2. Preliminaries and lemmas

This section collects the essential definitions and auxiliary results that underpin the convergence analysis in Section 3

Let H be a real Hilbert space. In what follows, we use the following notations:

• the symbols $\rightharpoonup$ stands for the weak convergence.

• the symbols $\to$ stands for the strong convergence.

Recall that a mapping $T : H \to H$ is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in H.$$

(2) firmly-nonexpansive if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \ \forall x, y \in H.$$

We note that if T is firmly-nonexpansive, then $I - T$ is also firmly-nonexpansive.

(3) L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in H.$$

A set-valued mapping $B : H \to 2^H$ is called monotone if for all $x, y \in H$

$$\langle u - v, x - y\rangle \geq 0, \ \forall u \in Bx \text{ and } v \in By.$$

For a set-valued mapping B, graph(B) is defined as graph$(B) := \{(x, u) \in H \times H : u \in B(x)\}$. A monotone mapping $B : H \to 2^H$ is said to be maximal if the graph(B) is not properly contained in the graph of any other monotone mapping. Let $B : H \to 2^H$ be a set-valued maximal monotone operator. The resolvent operator $J^B_\beta : H \to H$ associated with B is defined by

$$J^B_\beta(x) = (I + \beta B)^{-1}(x), \ \forall x \in H,$$

where $\beta > 0$. It is well known that the resolvent operator is single-valued and firmly non-expansive.

Since $A : H_1 \to H_2$ is a bounded linear operator and $\|A^*\| = \|A\|$, the composition $A^*(I - J^{B_2}_{\beta_n})A : H_1 \to H_1$ is well-defined and satisfies

$$\|A^*(I - J^{B_2}_{\beta_n})Au - A^*(I - J^{B_2}_{\beta_n})Av\| \leq \|A\|^2 \|u - v\| \quad \forall\, u, v \in H_1. \tag{2.17}$$

Lemma 2.1. *(Demiclosedness principle*[22]*) Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. If $x_n \rightharpoonup x \in C$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $x = Tx$.*

In order to study the SVIP, we recall some lemmas which are needed in our proof. We denote by $B^{-1}(0) = \{x \in H : 0 \in Bx\}$, $\mathcal{D}(T)$ the domain of T and $Fix(T)$ the fixed point set of T, that is, $Fix(T) = \{x \in H : x = Tx\}$.

Lemma 2.2. [6] *Let H be a real Hilbert space, $B : H \to 2^H$ be a set-valued maximal monotone mapping for each $x, y \in H$, each $z \in B^{-1}(0)$, and each $\beta > 0$; we have*

(i) $\langle x - J^B_\beta x, z - J^B_\beta x\rangle \leq 0$

(ii) $\|J^B_\beta x - J^B_\beta y\|^2 \leq \langle J^B_\beta x - J^B_\beta y, x - y\rangle$

(iii) $\|J^B_\beta x - z\|^2 \leq \|x - z\|^2 - \|J^B_\beta x - x\|^2$

Lemma 2.3. [16, 27] *Let H be a real Hilbert space, $B : H \to 2^H$ be a set-valued maximal monotone mapping. Thus,*

(i) J_β^B is a single-valued and firmly nonexpansive mapping for each $\beta > 0$;

(ii) $\mathcal{D}(J_\beta^B) = H$ and $Fix(J_\beta^B) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$;

(iii) $\|x - J_\beta^B x\| \leq \|x - J_\gamma^B x\|$ for all $0 < \beta \leq \gamma$ and for all $x \in H$;

(iv) Suppose that $B^{-1}(0) \neq \emptyset$. Then $\|x - J_\beta^B x\|^2 + \|J_\beta^B x - x^*\|^2 \leq \|x - x^*\|^2$ for each $x \in H$, each $x^* \in B^{-1}(0)$, and each $\beta > 0$.

(v) Suppose that $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_\beta^B x, J_\beta^B x - w\rangle \geq 0$ for each $x \in H$, each $w \in B^{-1}(0)$, and each $\beta > 0$.

The next lemma gives a crucial characterization of the solution sets of the SVIP and the fixed point sets of the resolvent operator.

Lemma 2.4. [16] *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $\beta > 0$, $\gamma > 0$, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone mappings. Given any $x^* \in H_1$.*

(i) If x^* is a solution of (SVIP), then $J_\beta^{B_1}(x^* - \gamma A^*(I - J_\beta^{B_2})Ax^*) = x^*$.

(ii) Suppose that $J_\beta^{B_1}(x^* - \gamma A^*(I - J_\beta^{B_2})Ax^*) = x^*$ and the solution set of (SVIP) is nonempty. Then x^* is a solution of (SVIP).

Lemma 2.5. [16] *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $\beta > 0$. Let $B : H_2 \to 2^{H_2}$ be a set-valued maximal monotone mapping. Define a mapping $T : H_1 \to H_1$ by $Tx := A^*(I - J_\beta^B)Ax$ for each $x \in H_1$. Then*

(i) $\|(I - J_\beta^B)Ax - (I - J_\beta^B)Ay\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in H_1$;

(ii) $\|A^*(I - J_\beta^B)Ax - A^*(I - J_\beta^B)Ay\|^2 \leq \|A\|^2 \cdot \langle Tx - Ty, x - y\rangle$ for all $x, y \in H_1$.

We also need the following tools in convergence analysis.

Lemma 2.6. [31] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real positive sequences such that*

$$a_{n+1} \leq (1 + c_n)a_n + b_n,\ n \geq 1.$$

If $\Sigma_{n=1}^{\infty} c_n < +\infty$ and $\Sigma_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to+\infty} a_n$ exists.

Lemma 2.7. *(Opial theorem* [30]*) Let C be a nonempty subset of a real Hilbert space H and $\{x_n\}$ be a sequence in H that satisfies the following properties:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ exists for each $x \in C$;

(ii) every sequential weak limit point of $\{x_n\}$ is in C.

Then $\{x_n\}$ converges weakly to a point in C.

3. Main results

We now state the proposed algorithm and establish its convergence properties. Let H_1 and H_2 be real Hilbert spaces. We define $A : H_1 \to H_2$ as a linear and bounded operator and A^* is the adjoint operator of A. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone operators. We denote the solution set of SVIP by Ω, where Ω is nonempty. We assume that

(I) Either $\alpha_n = 0, \forall\, n \geq 0$, or $-1 < d_1 \leq \alpha_n \leq \alpha_{n+1} \leq d_2 < 0$;

(II) $\{\beta_n\}$ is a sequence in $[\beta, \infty)$ for some $\beta > 0$;

(III) $\mu \in (0, 1)$, $\delta \in [0, 1)$, $\lambda_0 \geq 0$;

(IV) $\{a_n\}$ and $\{b_n\}$ satisfy $b_n = 1 - a_n$ with $\lim_{n\to\infty} a_n = 0$ (hence $\lim_{n\to\infty} b_n = 1$), and $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Algorithm 3.1. *Take arbitrarily $w_{-1}, w_{-2}, x_{-1}, x_0 \in H_1$. Set $n = 0$ and define*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) + \delta(1+\alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}), \tag{3.18}$$

$$y_n = J^{B_1}_{\beta_n}(w_n - \lambda_n A^*(I - J^{B_2}_{\beta_n})Aw_n), \tag{3.19}$$

$$x_{n+1} = y_n - \lambda_n(A^*(I - J^{B_2}_{\beta_n})Ay_n - A^*(I - J^{B_2}_{\beta_n})Aw_n), \tag{3.20}$$

where

$$\lambda_{n+1} = \begin{cases} \min\{(1+\gamma_n)\lambda_n, \dfrac{(a_n + b_n\mu)\,\|w_n - y_n\|}{\|A^*(I - J^{B_2}_{\beta_n})Aw_n - A^*(I - J^{B_2}_{\beta_n})Ay_n\|}\}, \\ \textit{if } A^*(I - J^{B_2}_{\beta_n})Aw_n \neq A^*(I - J^{B_2}_{\beta_n})Ay_n \\ (1+\gamma_n)\lambda_n, & \textit{otherwise} \end{cases} \tag{3.21}$$

Lemma 3.8. *Let the sequence $\{\lambda_n\}$ be defined by* (3.21)*. Then we have*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\frac{\mu}{\|A\|^2}, \lambda_0\right\} > 0, \tag{3.22}$$

and

$$\|A^*(I - J^{B_2}_{\beta_n})Aw_n - A^*(I - J^{B_2}_{\beta_n})Ay_n\| \leq \frac{a_n + b_n\mu}{\lambda_{n+1}}\,\|w_n - y_n\|. \tag{3.23}$$

Proof. We now show that $\lambda_n \geq \min\left\{\dfrac{\mu}{\|A\|^2}, \lambda_0\right\}$ for all $n \geq 0$ by induction. The base case $n = 0$ holds by assumption $\lambda_0 \geq 0$. Suppose the bound holds for n. In the non-trivial case of (3.21), by definition of λ_{n+1} and applying (2.17) to the denominator, we obtain

$$\lambda_{n+1} \geq \frac{(a_n + b_n\mu)\,\|w_n - y_n\|}{\|A^*(I - J^{B_2}_{\beta_n})Aw_n - A^*(I - J^{B_2}_{\beta_n})Ay_n\|} \geq \frac{a_n + b_n\mu}{\|A\|^2}.$$

Since $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1$, we have $a_n + b_n\mu \to \mu$ as $n \to \infty$, so $\liminf_{n\to\infty} \lambda_{n+1} \geq \dfrac{\mu}{\|A\|^2} > 0$. In the trivial case of (3.21), $\lambda_{n+1} = (1+\gamma_n)\lambda_n \geq \lambda_n$, so the lower bound is preserved. For the upper bound, from (3.21) we always have $\lambda_{n+1} \leq (1+\gamma_n)\lambda_n$, which gives

$$\lambda_n \leq \lambda_0 \prod_{k=0}^{n-1}(1+\gamma_k).$$

Since $\sum_{n=1}^{\infty}\gamma_n < \infty$, the infinite product $\prod_{k=0}^{\infty}(1+\gamma_k) < \infty$ converges, so $\{\lambda_n\}$ is bounded above. Being bounded both below and above, $\{\lambda_n\}$ converges and its limit satisfies (3.22).

To prove (3.23), if $A^*(I - J^{B_2}_{\beta_n})Aw_n \neq A^*(I - J^{B_2}_{\beta_n})Ay_n$, then by definition of λ_{n+1} in (3.21):

$$\lambda_{n+1} \leq \frac{(a_n + b_n\mu)\,\|w_n - y_n\|}{\|A^*(I - J^{B_2}_{\beta_n})Aw_n - A^*(I - J^{B_2}_{\beta_n})Ay_n\|},$$

and rearranging yields (3.23) directly. If $A^*(I - J^{B_2}_{\beta_n})Aw_n = A^*(I - J^{B_2}_{\beta_n})Ay_n$, the left-hand side of (3.23) is zero and the inequality holds trivially. □

Theorem 3.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Assume that conditions* (I)–(IV) *hold. Then $\{x_n\}$ converges weakly to a solution in Ω.*

Proof. Let $x^* \in \Omega$. Then $x^* \in B_1^{-1}(0)$ and $Ax^* \in B_2^{-1}(0)$. From Lemma 2.3(v), we have

$$\langle y_n - x^*, w_n - y_n - \lambda_n A^*(I - J^{B_2}_{\beta_n})Aw_n\rangle \geq 0. \tag{3.24}$$

Moreover, we have $J_{\beta_n}^{B_2}Ax^* = Ax^*$. It follows that

$$\begin{aligned}\langle A^*(I-J_{\beta_n}^{B_2})Ay_n, y_n - x^*\rangle &= \langle A^*(I-J_{\beta_n}^{B_2})Ay_n - A^*(I-J_{\beta_n}^{B_2})Ax^*, y_n - x^*\rangle \\ &= \langle (I-J_{\beta_n}^{B_2})Ay_n - (I-J_{\beta_n}^{B_2})Ax^*, Ay_n - Ax^*\rangle \\ &\geq \|(I-J_{\beta_n}^{B_2})Ay_n\|^2. \end{aligned} \tag{3.25}$$

By definition of x_{n+1}, we consider

$$\begin{aligned}&\|w_n - x^*\|^2 \\ =\ &\|w_n - y_n + y_n - x^*\|^2 \\ =\ &\|w_n - y_n\|^2 + \|y_n - x^*\|^2 + 2\langle w_n - y_n, y_n - x^*\rangle \\ =\ &\|w_n - y_n\|^2 + \|y_n - x_{n+1} + x_{n+1} - x^*\|^2 + 2\langle w_n - y_n, y_n - x^*\rangle \\ =\ &\|w_n - y_n\|^2 + \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x^*\|^2 + 2\langle y_n - x_{n+1}, x_{n+1} - x^*\rangle \\ &+2\langle w_n - y_n, y_n - x^*\rangle \\ =\ &\|w_n - y_n\|^2 + \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x^*\|^2 + 2\langle w_n - y_n, y_n - x^*\rangle \\ &+2\langle y_n - x_{n+1}, x_{n+1} - y_n + y_n - x^*\rangle \\ =\ &\|w_n - y_n\|^2 + \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x^*\|^2 \\ &+2\langle w_n - y_n, y_n - x^*\rangle - 2\langle y_n - x_{n+1}, y_n - x_{n+1}\rangle + 2\langle y_n - x_{n+1}, y_n - x^*\rangle \\ =\ &\|w_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x^*\|^2 \\ &+2\langle w_n - x_{n+1}, y_n - x^*\rangle \\ =\ &\|w_n - y_n\|^2 - \lambda_n^2\|A^*(I-J_{\beta_n}^{B_2})Ay_n - A^*(I-J_{\beta_n}^{B_2})Aw_n\|^2 \\ &+\|x_{n+1} - x^*\|^2 + 2\langle w_n - x_{n+1}, y_n - x^*\rangle. \end{aligned} \tag{3.26}$$

From (3.24) and (3.25), we obtain

$$\begin{aligned}&\langle w_n - x_{n+1}, y_n - x^*\rangle \\ =\ &\left\langle w_n - \left(y_n - \lambda_n\left(A^*(I-J_{\beta_n}^{B_2})Ay_n - A^*(I-J_{\beta_n}^{B_2})Aw_n\right)\right), y_n - x^*\right\rangle \\ =\ &\left\langle w_n - y_n - \lambda_n A^*(I-J_{\beta_n}^{B_2})Aw_n, y_n - x^*\right\rangle \\ &+\lambda_n\left\langle A^*(I-J_{\beta_n}^{B_2})Ay_n, y_n - x^*\right\rangle \\ \geq\ &\lambda_n^2\left\|(I-J_{\beta_n}^{B_2})Ay_n\right\|^2. \end{aligned} \tag{3.27}$$

From (3.26), (3.27) and Lemma 3.8, we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \lambda_n^2\|A^*(I-J_{\beta_n}^{B_2})Ay_n - A^*(I-J_{\beta_n}^{B_2})Aw_n\|^2 \\ &\quad -2\langle w_n - x_{n+1}, y_n - x^*\rangle \\ &\leq \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \frac{\lambda_n^2}{\lambda_{n+1}}(a_n + b_n\mu)^2\|w_n - y_n\|^2 \\ &\quad -2\lambda_n^2\|(I-J_{\beta_n}^{B_2})Ay_n\|^2 \\ &= \|w_n - x^*\|^2 - \left[1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n + b_n\mu)^2\right]\|w_n - y_n\|^2 \\ &\quad -2\lambda_n^2\|(I-J_{\beta_n}^{B_2})Ay_n\|^2 \end{aligned} \tag{3.28}$$

To handle the correction term in (3.18) and expose the double inertial structure, we introduce the auxiliary sequence

$$z_n = x_n + \delta(w_{n-1} - x_n), \tag{3.29}$$

it follows that

$$x_n = \frac{1}{1-\delta} z_n - \frac{\delta}{1-\delta} w_{n-1}. \tag{3.30}$$

From (3.29) and (3.30), we have

$$\begin{aligned} x_n - z_n &= \frac{1}{1-\delta} z_n - \frac{\delta}{1-\delta} w_{n-1} - z_n \\ &= \frac{1}{1-\delta}[z_n - \delta w_{n-1} - z_n + \delta z_n] \\ &= \frac{\delta}{1-\delta}(z_n - w_{n-1}) \\ &= \frac{\delta}{1-\delta}[(x_n + \delta(w_{n-1} - x_n)) - w_{n-1}] \\ &= \frac{\delta}{(1-\delta)}(1-\delta)[x_n - w_{n-1}] \\ &= \delta(x_n - w_{n-1}). \end{aligned} \tag{3.31}$$

From (3.29), we have

$$z_{n-1} = x_{n-1} + \delta(w_{n-2} - x_{n-1}). \tag{3.32}$$

By definition of the sequence $\{w_n\}$, we obtain

$$\begin{aligned} w_n &= x_n + \alpha_n(x_n - x_{n-1}) + \delta(1+\alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}) \\ &= x_n + \delta(w_{n-1} - x_n) + \alpha_n[(x_n + \delta(w_{n-1} - x_n)) - (x_{n-1} + \delta(w_{n-2} - x_{n-1}))]. \end{aligned} \tag{3.33}$$

Replacing (3.29) and (3.32) in (3.33), we obtain

$$w_n = z_n + \alpha_n(z_n - z_{n-1}). \tag{3.34}$$

We see that,

$$\|z_n - w_{n-1}\|^2 = (1-\delta)^2\|x_n - w_{n-1}\|^2. \tag{3.35}$$

From (3.30) and (3.35), we get

$$\begin{aligned} \|x_n - x^*\|^2 &= \|(\frac{1}{1-\delta} z_n - \frac{\delta}{1-\delta} w_{n-1}) - x^*\|^2 \\ &= \|\frac{1}{1-\delta}(z_n - x^*) - \frac{\delta}{1-\delta}(w_{n-1} - x^*)\|^2 \\ &= \frac{1}{1-\delta}\|z_n - x^*\|^2 - \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 + \frac{\delta}{(1-\delta)^2}\|z_n - w_{n-1}\|^2 \\ &= \frac{1}{1-\delta}\|z_n - x^*\|^2 - \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 + \delta\|x_n - w_{n-1}\|^2. \end{aligned} \tag{3.36}$$

Adding $\|x_n - x^*\|^2$ in equation (3.28) and using (3.36), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \\
\leq\ & \|w_n - x^*\|^2 - \|x_n - x^*\|^2 - \left[1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n + b_n\mu)^2\right]\|w_n - y_n\|^2 \\
& -2\lambda_n^2\|(I - J_{\beta_n}^{B_2})Ay_n\|^2 \\
=\ & \|w_n - x^*\|^2 - \frac{1}{1-\delta}\|z_n - x^*\|^2 + \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 - \delta\|x_n - w_{n-1}\|^2 \\
& - \left[1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n + b_n\mu)^2\right]\|w_n - y_n\|^2 - 2\lambda_n^2\|(I - J_{\beta_n}^{B_2})Ay_n\|^2. \qquad (3.37)
\end{aligned}$$

From (3.34), we obtain

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|z_n + \alpha_n(z_n - z_{n-1}) - x^*\|^2 \\
&= (1+\alpha_n)\|z_n - x^*\|^2 - \alpha_n\|z_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|z_n - z_{n-1}\|^2. \qquad (3.38)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|w_{n-1} - x^*\|^2 &= (1+\alpha_{n-1})\|z_{n-1} - x^*\|^2 - \alpha_{n-1}\|z_{n-2} - x^*\|^2 \\
&\quad +\alpha_{n-1}(1+\alpha_{n-1})\|z_{n-1} - z_{n-2}\|^2. \qquad (3.39)
\end{aligned}$$

Replacing (3.38), (3.39) in (3.37), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \\
\leq\ & (1+\alpha_n)\|z_n - x^*\|^2 - \alpha_n\|z_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|z_n - z_{n-1}\|^2 \\
& -\frac{1}{1-\delta}\|z_n - x^*\|^2 + \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 - \delta\|x_n - w_{n-1}\|^2 \\
& - \left[1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n + b_n\mu)^2\right]\|w_n - y_n\|^2 - 2\lambda_n^2\|(I - J_{\beta_n}^{B_2})Ay_n\|^2 \\
=\ & (\alpha_n - \frac{\delta}{1-\delta})\|z_n - x^*\|^2 - \alpha_n\|z_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|z_n - z_{n-1}\|^2 \\
& +\frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 - \delta\|x_n - w_{n-1}\|^2 \\
& -\frac{1}{1-\delta}\|z_n - x^*\|^2 + \frac{\delta}{1-\delta}\|w_{n-1} - x^*\|^2 - \delta\|x_n - w_{n-1}\|^2 \\
& - \left[1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n + b_n\mu)^2\right]\|w_n - y_n\|^2 - 2\lambda_n^2\|(I - J_{\beta_n}^{B_2})Ay_n\|^2 \\
=\ & (\alpha_n - \frac{\delta}{1-\delta})\|z_n - x^*\|^2 - \alpha_n\|z_{n-1} - x^*\|^2 + \alpha_n(1+\alpha_n)\|z_n - z_{n-1}\|^2 \\
& +\frac{\delta}{1-\delta}[(1+\alpha_{n-1})\|z_{n-1} - x^*\|^2 - \alpha_{n-1}\|z_{n-2} - x^*\|^2 + \alpha_{n-1}(1+\alpha_{n-1})\|z_{n-1} - z_{n-2}\|^2] \\
& (3.40)\|x_n - w_{n-1}\|^2 - \left[1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n + b_n\mu)^2\right]\|w_n - y_n\|^2 - 2\lambda_n^2\|(I - J_{\beta_n}^{B_2})Ay_n\|^2.
\end{aligned}$$

From (3.40), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
\leq\ &\|x_n-x^*\|^2+(\frac{\delta(1+\alpha_{n-1})}{1-\delta}-\alpha_n)\|z_{n-1}-x^*\|^2\\
&+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2\\
&-\delta\|x_n-w_{n-1}\|^2-\left[1-\frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n+b_n\mu)^2\right]\|w_n-y_n\|^2-2\lambda_n^2\|(I-J_{\beta_n}^{B_2})Ay_n\|^2\\
=\ &\|x_n-x^*\|^2+(\frac{\delta}{1-\delta}+\frac{\delta\alpha_{n-1}}{1-\delta}-\alpha_n)\|z_{n-1}-x^*\|^2\\
&+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2\\
&-\delta\|x_n-w_{n-1}\|^2-\left[1-\frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n+b_n\mu)^2\right]\|w_n-y_n\|^2-2\lambda_n^2\|(I-J_{\beta_n}^{B_2})Ay_n\|^2. \qquad (3.41)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-1}-x^*\|^2-\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
\leq\ &\|x_n-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-\left[1-\frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n+b_n\mu)^2\right]\|w_n-y_n\|^2\\
&-2\lambda_n^2\|(I-J_{\beta_n}^{B_2})Ay_n\|^2.
\end{aligned}$$

Since $\alpha_{n-1}\leq\alpha_n$, we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2-(\alpha_n-\frac{\delta}{1-\delta})\|z_n-x^*\|^2-\frac{\delta\alpha_n}{1-\delta}\|z_{n-1}-x^*\|^2-\alpha_n(1+\alpha_n)\|z_n-z_{n-1}\|^2\\
\leq\ &\|x_n-x^*\|^2-(\alpha_{n-1}-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2-\delta\|x_n-w_{n-1}\|^2\\
&+\frac{\delta\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2-\left[1-\frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n+b_n\mu)^2\right]\|w_n-y_n\|^2\\
&-2\lambda_n^2\|(I-J_{\beta_n}^{B_2})Ay_n\|^2\\
=\ &\|x_n-x^*\|^2-(\alpha_{n-1}-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2\\
&-\alpha_{n-1}(1+\alpha_{n-1})\|z_{n-1}-z_{n-2}\|^2+\frac{\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2\\
&-\delta\|x_n-w_{n-1}\|^2-\left[1-\frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n+b_n\mu)^2\right]\|w_n-y_n\|^2-2\lambda_n^2\|(I-J_{\beta_n}^{B_2})Ay_n\|^2.
\end{aligned}$$

We define,

$$\begin{aligned}
\rho_n\ =\ &\|x_n-x^*\|^2-(\alpha_{n-1}-\frac{\delta}{1-\delta})\|z_{n-1}-x^*\|^2-\frac{\delta\alpha_{n-1}}{1-\delta}\|z_{n-2}-x^*\|^2\\
&-\alpha_{n-1}(1+\alpha_{n-1})\|z_{n-1}-z_{n-2}\|^2. \qquad (3.42)
\end{aligned}$$

From (3.42), we get

$$\begin{aligned}\rho_{n+1} &\leq \rho_n + \frac{\alpha_{n-1}(1+\alpha_{n-1})}{1-\delta}\|z_{n-1}-z_{n-2}\|^2 - \delta\|x_n - w_{n-1}\|^2 \\ &\quad - \left[1 - \frac{\lambda_n^2}{\lambda_{n+1}^2}(a_n + b_n\mu)^2\right]\|w_n - y_n\|^2 - 2\lambda_n^2\|(I - J_{\beta_n}^{B_2})Ay_n\|^2. \end{aligned} \tag{3.43}$$

From Lemma 3.8, we have $\lim_{n\to\infty}\lambda_n = \lambda \geq \min\{\frac{\mu}{\|A\|^2}, \lambda_0\} \geq 0$, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 1$ and $\mu \in (0,1)$. Also since, $\alpha_n \in (-1,0]$ and $\delta \in [0,1)$, from (3.43) and Lemma 2.6, $\lim_{n\to\infty}\rho_n$ exists and $\{\rho_n\}$ is bounded. From (3.42), we obtain $\{x_n\}$ is also bounded.

Since $\alpha \in (-1,0]$, we consider two cases as follows:

Case I Set $\alpha_n = 0$. From (3.33), we get

$$\begin{aligned} w_n &= x_n + \alpha_n(x_n - x_{n-1}) + \delta(1+\alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}) \\ &= x_n + \delta(w_{n-1} - x_n). \end{aligned} \tag{3.44}$$

If $\delta \in (0,1)$, in (3.43), we obtain

$$\lim_{n\to\infty}\|x_n - w_{n-1}\| = 0. \tag{3.45}$$

Thus, from (3.44) and (3.45),

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\delta\|w_{n-1} - x_n\| = 0. \tag{3.46}$$

If $\delta = 0$, from (3.29) we have $z_n = x_n$, we obtain

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|x_n + \delta(w_{n-1} - x_n) - x_n\| = 0. \tag{3.47}$$

It follows that $\lim_{n\to\infty}\|w_n - x_n\| = 0$, if $\delta \in [0,1)$.

Case II Set $-1 < d_1 \leq \alpha_n \leq \alpha_{n+1} \leq d_2 < 0$ in (3.43).

If $\delta \in (0,1)$ in (3.43), we have $\lim_{n\to\infty}\|z_{n-1} - x_{n-2}\| = 0$ and $\lim_{n\to\infty}\|x_n - w_{n-1}\| = 0$.

If $\delta = 0$ in (3.43), we also have $\lim_{n\to\infty}\|x_n - w_{n-1}\| = 0$.

Thus, from $\delta \in [0,1)$ we obtain

$$\lim_{n\to\infty}\|x_n - w_{n-1}\| = 0. \tag{3.48}$$

From (3.29) and (3.48), we get

$$\begin{aligned}\lim_{n\to\infty}\|z_n - x_n\| &= \lim_{n\to\infty}\|x_n + \delta(w_{n-1} - x_n) - x_n\| \\ &= \delta\lim_{n\to\infty}\|w_{n-1} - x_n\| \\ &= 0. \end{aligned} \tag{3.49}$$

From (3.34), we get

$$\lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}|\alpha_n|\|z_n - z_{n-1}\| = 0. \tag{3.50}$$

Thus, from (3.49) and (3.50), we get

$$\lim_{n\to\infty}\|w_n - x_n\| \leq \lim_{n\to\infty}\|w_n - z_n\| + \lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.51}$$

Therefore, from **Case I** and **II**, we can conclude that

$$\lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.52}$$

From (3.43), we have

$$\lim_{n\to\infty}\|y_n - w_n\| = 0. \tag{3.53}$$

From (3.52) and (3.53), we get

$$\lim_{n\to\infty}\|y_n - x_n\| \leq \lim_{n\to\infty}\|y_n - w_n\| + \lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.54}$$

From (3.43), we obtain

$$\lim_{n\to\infty}\|Ay_n - J^{B_2}_{\beta_n}Ay_n\| = 0, \tag{3.55}$$

Using Lemma 2.3(iii) and (3.55), we have

$$\lim_{n\to\infty}\|Ay_n - J^{B_2}_{\beta}Ay_n\| \leq \lim_{n\to\infty}\|Ay_n - J^{B_2}_{\beta_n}Ay_n\| = 0. \tag{3.56}$$

From (3.54) and (3.55), we have

$$\begin{aligned}\|Ax_n - J^{B_2}_{\beta_n}Ax_n\| &\leq \|(Ax_n - J^{B_2}_{\beta_n}Ax_n) - (Ay_n - J^{B_2}_{\beta_n}Ay_n)\| + \|Ay_n - J^{B_2}_{\beta_n}Ay_n\| \\ &\leq \|A\|\|x_n - y_n\| + \|Ay_n - J^{B_2}_{\beta_n}Ay_n\| \\ &\to 0. \end{aligned} \tag{3.57}$$

Also, from (3.57) and Lemma 2.3(iii), we have

$$\lim_{n\to\infty}\|Ax_n - J^{B_2}_{\beta}Ax_n\| \leq \lim_{n\to\infty}\|Ax_n - J^{B_2}_{\beta_n}Ax_n\| = 0. \tag{3.58}$$

Since $J^{B_1}_{\beta_n}$ is nonexpansive and from (3.54), (3.55), we obtain

$$\begin{aligned}\|y_n - J^{B_1}_{\beta_n}y_n\| &= \|J^{B_1}_{\beta_n}(x_n - \lambda_n A^*(I - J^{B_2}_{\beta_n})Ax_n) - J^{B_1}_{\beta_n}y_n\| \\ &\leq \|x_n - \lambda_n A^*(I - J^{B_2}_{\beta_n})Ax_n - y_n\| \\ &\leq \|x_n - y_n\| + \|\lambda_n A^*(I - J^{B_2}_{\beta_n})Ax_n\| \\ &\leq \|x_n - y_n\| + \lambda_n\|A^*\|\|(I - J^{B_2}_{\beta_n})Ax_n\| \\ &\to 0. \end{aligned} \tag{3.59}$$

Also, from (3.59) and Lemma 2.3(iii), we have

$$\lim_{n\to\infty}\|y_n - J^{B_1}_{\beta}y_n\| \leq \lim_{n\to\infty}\|y_n - J^{B_1}_{\beta_n}y_n\| = 0. \tag{3.60}$$

From (3.54) and (3.60), we have

$$\begin{aligned}\|x_n - J^{B_1}_{\beta}x_n\| &\leq \|x_n - J^{B_1}_{\beta}x_n - y_n + y_n - J^{B_1}_{\beta}y_n + J^{B_1}_{\beta}y_n\| \\ &= \|(x_n - J^{B_1}_{\beta}x_n) - (y_n - J^{B_1}_{\beta}y_n) + (y_n - J^{B_1}_{\beta}y_n)\| \\ &\leq \|(I - J^{B_1}_{\beta})x_n - (I - J^{B_1}_{\beta})y_n\| + \|y_n - J^{B_1}_{\beta}y_n\| \\ &\leq \|x_n - y_n\| + \|y_n - J^{B_1}_{\beta}y_n\| \\ &\to 0. \end{aligned} \tag{3.61}$$

We see that the sequence $\{x_n\}$ is bounded. Let z^* be a weak cluster point of $\{x_n\}$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z^*$. Since A is a bounded linear operator, we have $Ax_{n_k} \rightharpoonup Az^*$. By (3.58), (3.61), Lemma 2.1 and Lemma 2.3(ii), we have $z^* \in \Omega$. By Lemma 2.7, we conclude that the sequence $\{x_n\}$ converges weakly to a point in Ω. This completes the proof. □

Remark 3.1. *The terminology double inertial is justified by the decomposition revealed in* (3.34). *Defining the auxiliary sequence*

$$z_n = x_n + \delta(w_{n-1} - x_n),$$

the update (3.19) *is equivalently written as*

$$w_n = z_n + \alpha_n(z_n - z_{n-1}),$$

showing that the algorithm applies two successive inertial extrapolation steps: the first produces z_n from x_n via the correction term controlled by δ, and the second produces w_n from z_n via the momentum parameter α_n. When $\delta = 0$, the correction step vanishes and the scheme reduces to a standard single inertial method.

Remark 3.2. *Condition (I) restricts the inertial parameter to $\alpha_n \in (-1, 0]$, in contrast to classical inertial methods which require $\alpha_n \geq 0$. The non-positive range is necessitated by the presence of the correction term $\delta \in [0, 1)$ in* (3.18)*: allowing negative α_n ensures the combined effect of the inertial and correction terms remains controlled, which is essential for the boundedness argument and the convergence analysis in Theorem 3.1. When $\delta = 0$ and $\alpha_n = 0$, Algorithm 3.1 recovers the non-inertial baseline.*

4. Applications

The corollary of Theorem 3.1 is derived by specializing the SVIP to particular operator choices.

4.1. **Application to the split feasibility problem.**

Consider the following split feasibility problem (SFP) suggested by Censor and Elfving [13].

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{4.62}$$

where C and Q are closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

The definition of the subdifferential ∂g for a proper lower semicontinuous convex function $g : H \to (-\infty, \infty)$ is defined by

$$\partial g(x) = \{z \in H : g(x) - g(y) \leq \langle z, x - y\rangle \ \forall y \in H\},$$

for all $x \in H$. Now, let C be a nonempty closed convex subset of H, and let i_C be the indicator function of C which is defined as follows:

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

Further, the normal cone $N_C u$ of C at $u \in C$ is defined as

$$N_C u = \{z \in H : \langle z, v - u\rangle \leq 0, \ \forall v \in C\}.$$

It is known that i_C is a proper, lower semicontinuous and convex function on H, hence, the subdifferential ∂i_C is a maximal monotone operator. Thus, the resolvent $J_\beta^{\partial i_C}$ for each $\beta > 0$ can be defined as

$$J_\beta^{\partial i_C}(x) = (I + \beta \partial i_C)^{-1}(x)$$

for all $x \in H$.

It is known that for any $x \in C$, $\partial i_C x = N_C x$. Putting $y = J_\beta^{\partial i_C} x = (I + \beta \partial i_C)^{-1}(x)$, $\beta > 0$, we obtain

$$\begin{aligned} x \in y + \beta \partial i_C(y) &\Leftrightarrow x - y \in \beta \partial i_C y \\ &\Leftrightarrow \langle x - y, z - y\rangle \leq 0 \ \forall z \in C \\ &\Leftrightarrow y = P_C x. \end{aligned}$$

Consequently, we have the following results:

Algorithm 4.1. *Let $\alpha_n \in (-1,0]$, $\delta \in [0,1)$, $\mu \in (0,1)$, $\lambda_0 \geq 0$. Take arbitrarily $w_{-1}, w_{-2}, x_{-1}, x_0 \in H_1$. Set $n = 0$ and define*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) + \delta(1+\alpha_n)(w_{n-1} - x_n) - \alpha_n\delta(w_{n-2} - x_{n-1}), \tag{4.63}$$

$$y_n = P_C(w_n - \lambda_n A^*(I - P_Q)Aw_n), \tag{4.64}$$

$$x_{n+1} = y_n - \lambda_n(A^*(I - P_Q)Ay_n - A^*(I - P_Q)Aw_n), \tag{4.65}$$

where

$$\lambda_{n+1} = \begin{cases} \min\{(1+\gamma_n)\lambda_n, \dfrac{(a_n + b_n\mu)\,\|w_n - y_n\|}{\|A^*(I-P_Q)Aw_n - A^*(I-P_Q)Ay_n\|}\}, \\ \textit{if } A^*(I-P_Q)Aw_n \neq A^*(I-P_Q)Ay_n \\ (1+\gamma_n)\lambda_n, \qquad\qquad \textit{otherwise} \end{cases} \tag{4.66}$$

Consequently, Theorem 3.1 reduces to this following convergence result for split feasibility problem (4.62).

Corollary 4.1. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let the sequence x_n be generated by Algorithm 4.1. Then the sequence x_n converges weakly to a solution of the SFP.*

4.2. **Numerical experiments to an image recovery.** In this section, we present numerical experiments to an image recovery and compare with Algorithm 3.1 of Wang et al. [37] and Algorithm 3.1 of Tang and Gibali [33], the works which study the split feasibility. Let $C = [0,255]^D$ with $D = M \times N$ where M is the pixels of width and N is the pixels of height of color image. Consider the constrained minimization problem:

$$\min_{x \in C} \|Ax - y\|_2. \tag{4.67}$$

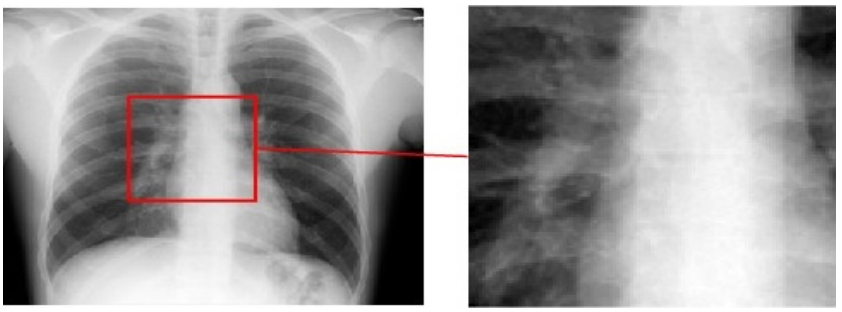
Therefore, the problem (4.62) can be reduced to (4.67) when $Q = \{y\}$ and $C = [0,255]^D$. We will compare the following methods with $w_{-1}, w_{-2}, x_0 = (0,0,0,...,0) \in \mathbb{R}^N$ and $x_{-1} = (1,1,1,...,1) \in \mathbb{R}^N$. We set parameters by

Algorithm 4.1: $\delta = 0.001$, $\mu = 0.9$, $\lambda_0 = 2$, $\alpha_n = -\frac{1}{10(n+1)}$, $\gamma_n = \frac{1}{n^2}$, $a_n = \frac{1}{n+1}$ and $b_n = 1 - a_n$.

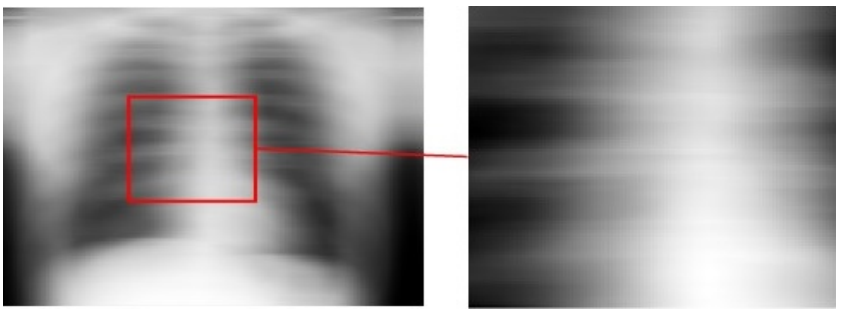
Algorithm 3.1 of Wang et al. [37]: $\lambda_0 = 1$, $\mu = 0.5$, $\alpha_n = 1 - \frac{1}{10^n}$, $\beta_n = 0.1 - \frac{1}{1000+n}$, $\theta_n = 0.45 - \frac{1}{1000+n}$, $\mu_n = \frac{1}{n^2}$ and $p_n = \frac{1}{n^2}$.

Algorithm 3.1 of Tang and Gibali [33]: $\rho = 0.3$ and $\alpha = 0.5$.

We consider motion blur with motion length of 45 pixels and motion orientation $180°$, the image size 289×349 for grey image.



(A)



(B)

FIGURE 1. (A) The original image and (B) Motion blurred image

To measure the restored images, we use the peak-signal-to-noise ratio (PSNR) [34] defined by

$$PSNR = 10\log_{10}\left(\frac{255^2}{MSE}\right) \tag{4.68}$$

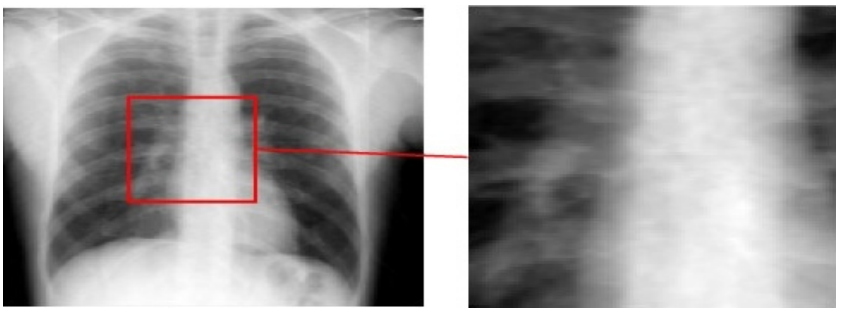
where MSE$= \|x_n - x\|^2$ with x is an original image. We also use structural similarity index measure (SSIM) [36] for measuring the similarity between the original and blurred images. From definitions, it is clear that the high PSNR and SSIM values show the quality of restored images. We obtain numerical results following Table 1

TABLE 1. Comparison of PSNR, SSIM, and MSE of the restored images for 500 iterations. Higher PSNR and SSIM values and lower MSE values indicate better restoration quality.

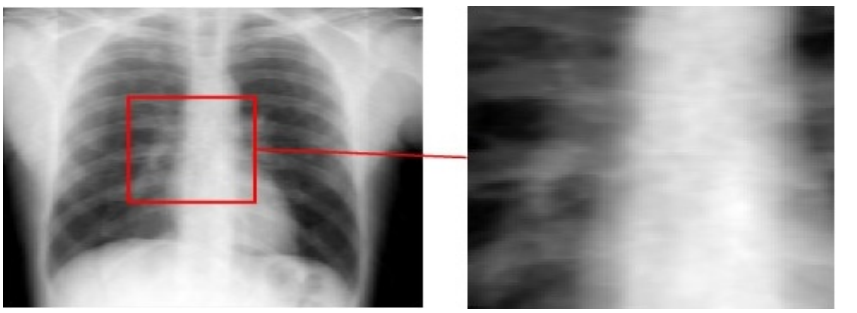
Algorithms	PSNR	SSIM	MSE
Algorithm 4.1	33.4686	0.9565	1.3029×10^{-5}
[37, Algorithm 3.1]	31.9950	0.9449	1.3779×10^{-5}
[33, Algorithm 3.1]	30.4856	0.9299	2.2739×10^{-5}

From Table 1, we observe that our Algorithm 4.1 has more efficient than others since PSNR, SSIM and MSE values of Algorithm4.1 have the highest number in the experiment for 500 iterations.

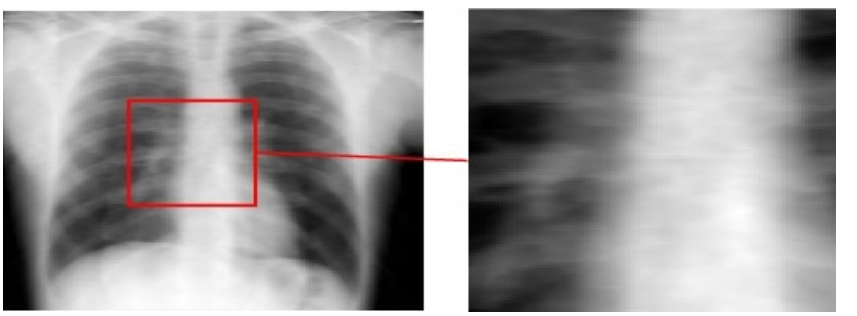
We next show the figures of restored images.



(A) The restored images by Algorithm 4.1, PSNR=33.4686

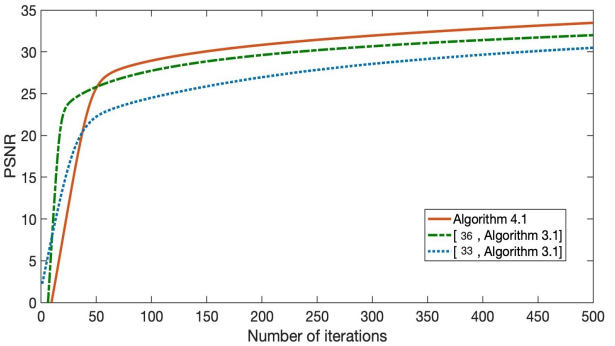


(B) The restored images by Algorithm 3.1 of Wang et al. [37], PSNR=31.9950

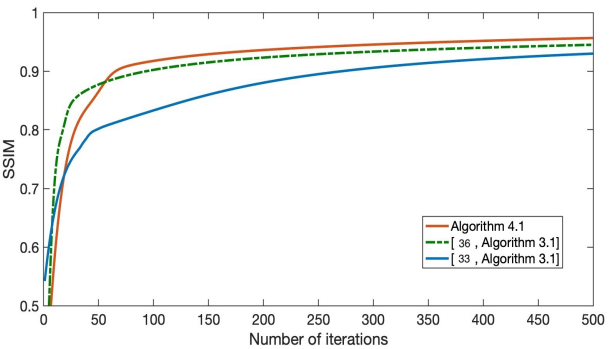


(C) The restored images by Algorithm 3.1 of Tang and Gibali [33],PSNR=30.4856

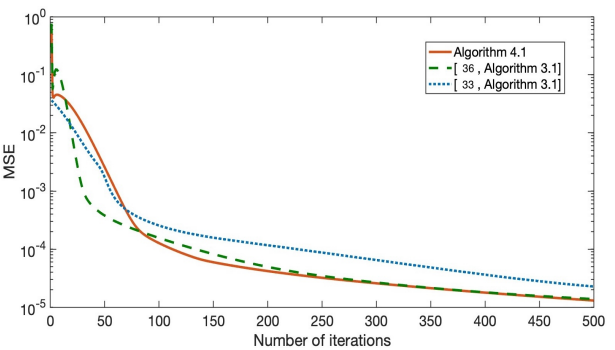
FIGURE 2. (A),(B), and (C) are the restored images by Algorithm 4.1, Algorithm 3.1 of Wang et al. [37] and Algorithm 3.1 of Tang and Gibali [33], respectively



(A) PSNR plotting



(B) SSIM plotting



(C) MSE plotting

FIGURE 3. Convergence plots of (A) PSNR, (B) SSIM, and (C) MSE over 500 iterations for Algorithm 4.1, Algorithm 3.1 of Wang et al. [37] and Algorithm 3.1 of Tang and Gibali [33]

In Figures 1, 2 and 3, it is shown that Algorithm 4.1 outperforms the other algorithms in terms of PSNR, SSIM and MSE for the particular problem considered.

5. Conclusions

This work has presented a novel splitting scheme for the SVIP in real Hilbert spaces. Two acceleration mechanisms are embedded in the extrapolated point w_n: a momentum term weighted by $\alpha_n \in (-1, 0]$ and a correction step governed by $\delta \in [0, 1)$. The step size λ_n is selected at each iteration through a self-adaptive rule, making prior estimation of $\|A\|$ unnecessary. Theorem 3.1 guarantees weak convergence of $\{x_n\}$ to a solution of the SVIP under Conditions (I)–(IV) whenever $\Omega \neq \emptyset$. Specializing to the split feasibility problem recovers Algorithm 4.1 and its convergence guarantee (Corollary 4.1). Numerical evidence from motion-blurred medical image reconstruction corroborates the theoretical findings. After 500 iterations, Algorithm 4.1 attains PSNR = 33.4686, SSIM = 0.9565, and MSE = 1.3029×10^{-5}, improving over Wang et al. [37] (PSNR = 31.9950, SSIM = 0.9449, MSE = 1.3779×10^{-5}) and Tang and Gibali [33] (PSNR = 30.4856, SSIM = 0.9299, MSE = 2.2739×10^{-5}). These margins confirm that the combined double inertial and correction mechanism yields meaningfully better restoration quality in practice.

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