CARPATHIAN J. MATH.
Volume **43** (2027), No. 1,
Pages 31-56

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2027.01.03

Ranking of Scenario-Based Portfolios via Hybrid Robustness-TOPSIS Approach

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ABSTRACT. Portfolio optimization is inherently a multi-criteria decision-making problem, yet conventional models often fail to address the profound uncertainty of financial markets. Deterministic approaches, by optimizing for nominal performance, can create brittle portfolios that are excessively risky, while overly conservative robust strategies often lead to disappointing returns by sacrificing too much upside. To achieve an optimal balance between performance and resilience, our framework systematically identifies the most robust portfolio from a feasible set of alternatives that satisfy a user-defined threshold for nominal performance. The methodology integrates scenario-based analysis with a custom multi-criteria scoring technique to generate scenario-wise priority rankings. These rankings are then used to select the final portfolio, ensuring resilience under both nominal and adverse market conditions. Empirical experiments demonstrate that the proposed approach yields a balanced portfolio that outperforms those derived from conservative robustness methods while maintaining strong nominal performance. Furthermore, sensitivity analysis confirms the stability of the rankings against variations in criteria weights, reinforcing the reliability of the results. By explicitly managing the trade-off between performance and resilience, this framework offers a transparent, stable, and practical tool for modern portfolio decision-making.

1. INTRODUCTION

The selection of an investment portfolio is a fundamental financial decision involving the allocation of capital across assets under uncertain future returns [7, 11, 6]. Markowitz's seminal work [12] laid the theoretical foundation for modern portfolio theory (MPT), which provides a systematic framework for asset allocation. MPT quantifies the trade-off between risk and return through the mean–variance (MV) model, where portfolio variance represents risk and expected return denotes reward [18, 1, 11, 20].

The MV model is an important starting point, but its focus on only two criteria is often too limited for real-world investment decisions. In practice, portfolio selection requires comparing multiple alternatives across several objectives that may conflict with one another. This motivates the use of multi-criteria decision-making (MCDM) methods, which provide a systematic way to rank portfolios while explicitly incorporating the decision-maker's stated preferences [23]. Because these preferences must be specified through inputs such as the relative importance of each criterion, acceptable levels of downside risk, and the desired trade-off between return and risk, some subjectivity is unavoidable. These choices reflect individual risk tolerance and investment goals, and they can lead to different rankings even when the same data are used. Nonetheless, MCDM frameworks rely on clear mathematical procedures that make these preference assumptions explicit and traceable, thereby keeping the influence of subjectivity limited and transparent.

The use of MCDM in portfolio selection was pioneered in several foundational studies. Hababou and Martel [9], for instance, applied the PROMETHEE II method to formalize

Received: 26.08.2025. In revised form: 20.12.2025. Accepted: 10.02.2026
2020 *Mathematics Subject Classification.* 90C11, 90C27, 90C90.
Key words and phrases. *Multi-criteria decision-making; TOPSIS; Portfolio optimization, Robustness; Uncertain multi-criteria optimization .*
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the selection process through a systematic methodology. Subsequently, Bouri et al. [4] expanded the problem's scope by incorporating non-financial criteria, such as investor attitudes toward solvency and liquidity, thereby underscoring the need for specialized techniques to handle the multifaceted nature of modern portfolio management. Xidonas et al. [21] proposed an integrated multi-criteria framework for portfolio construction and selection that explicitly accounts for the multidimensional nature of the decision problem. A key benefit of their approach is that it enables systematic comparison among the resulting portfolios. Yodmun and Witayakiattilerd [22] studied stock portfolio selection using a fuzzy-based scheme. They first grouped stocks by industry sector and then ranked them using fuzzy quantitative weighting, followed by a fuzzy analytic hierarchy process (FAHP)–TOPSIS procedure to prioritize stocks within each group. Within finance, recent literature showcases a variety of hybrid MCDM models for ranking investments. For example, Vásquez et al [19]. proposed a combined AHP-TOPSIS framework to rank portfolios using subjectively weighted criteria. Further advancements in the MCDM algorithms themselves seek to create more robust and discerning priority indices. While various MCDM models have been applied to finance, recent advancements in hybrid methodologies offer even greater potential for decision-making. Notably, in the field of risk evaluation, Pintelon et al. [15] developed a novel hybrid method that uniquely combines objective entropy weighting with the strengths of three distinct techniques: TOPSIS for defining ideal solutions, Grey Relational Analysis (GRA) for calculating relational similarity, and Evaluation based on Distance from Average Solution (EDAS) for the final ranking. This sophisticated synthesis of techniques has been shown to produce more reliable rankings than simpler models. The approach is powerful because it provides a structured, objective, and mathematically rigorous way to rank complex alternatives based on their performance across multiple conflicting criteria. Its effectiveness has been validated in other critical applications, such as the risk assessment framework for medical device design proposed by Phokee et al. [16], confirming its status as a robust and versatile decision-making tool.

Building upon these recent advancements, this paper addresses two key opportunities for development. First, existing frameworks, while sophisticated, are fundamentally deterministic and thus require extension to effectively model the stochastic nature of financial markets and assess portfolio resilience. Second, the current literature has prioritized the algorithmic development of these models over a formal analysis of their theoretical properties. For instance, while the hybrid method of Pintelon et al. [15] is procedurally robust, a theoretical examination of its ranking characteristics remains an open research question.

This paper introduces a novel hybrid robustness-MCDM framework that addresses both of these opportunities. Our methodology advances the state-of-the-art through three primary contributions. First, we generalize the deterministic hybrid MCDM process into a stochastic, scenario-based framework to explicitly model market uncertainty. Second, we introduce a robust aggregation rule that selects a portfolio based on its worst-case appraisal score, thereby optimizing for resilience. Third, we provide the first formal theoretical analysis of this extended framework, establishing its key properties to ensure its analytical soundness. By integrating scenario-based evaluation with robust selection and theoretical validation, our framework provides a transparent, stable, and practical tool for modern portfolio selection.

The remainder of this paper is organized as follows. Section 2 details the mathematical formulation of our hybrid framework. Section 3 provides a theoretical analysis of its properties. Section 4 describes the experimental setup and data. Section 5 presents and

discusses the compelling results of our empirical study, and Section 6 concludes with a summary of our findings.

2. Mathematical Model Formulation

We begin this section by recalling the robustness concept that will be adopted to introduce a new hybrid robustness-MCDM method.

2.1. **Uncertain single-objective optimization.** Let $X \subseteq \mathbb{R}^n$ denote the set of feasible portfolio alternatives, and let $\mathcal{U} = \{s_0, s_1, \ldots, s_{p-1}\}$ represent a finite set of p market scenarios capturing different states of uncertainty. An uncertain single-objective optimization problem, denoted $\mathcal{P}(\mathcal{U})$, is defined as a collection of deterministic subproblems $\{\mathcal{P}(s_q) : s_q \in \mathcal{U}\}$, where each subproblem corresponds to a scenario $s_q \in \mathcal{U}$:

$$\mathcal{P}(s_q): \quad \min_{x \in X} f(x, s_q), \tag{2.1}$$

where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $f(x, s_q)$ denotes the objective function value of an alternative x under scenario s_q.

Several solution concepts have been introduced for the uncertain single-objective optimization problem $\mathcal{P}(\mathcal{U})$, each depending on a specific problem-solving approach. In the context of portfolio selection, it is desirable to identify a solution that performs well under the worst-case scenario while maintaining near-optimal performance in a designated nominal scenario. Building on this idea, Schöbel [17] introduced the concept of light robustness for the problem $\mathcal{P}(\mathcal{U})$, as defined below.

Definition 2.1 (Light robust optimality [17])**.** *Given the uncertain single-objective optimization problem $\mathcal{P}(\mathcal{U})$, together with a nominal scenario $\hat{s} \in \mathcal{U}$ and a tolerance parameter $\epsilon > 0$, a solution $x^* \in X$ is said to be a* lightly robust optimal solution *if it solves:*

$$\min_{x \in X} \sup_{s \in \mathcal{U}} f(x, s), \tag{2.2}$$

$$\textit{subject to} \quad f(x, \hat{s}) \leq \min_{\bar{x} \in X} f(\bar{x}, \hat{s}) + \epsilon. \tag{2.3}$$

2.2. **Hybrid robustness-MCDM portfolio evaluation framework.** The proposed evaluation framework is a substantial extension of the hybrid MCDM methodology introduced by Pintelon et al. [15]. In their method, TOPSIS provides the core ranking structure, and entropy-based objective weighting, Grey Relational Analysis (GRA), and EDAS-style deviation scoring are incorporated as supporting steps. However, the original framework was developed for deterministic settings using a single static performance matrix. This single-matrix formulation is well suited to reliability engineering problems, but it does not fully capture the uncertainty and changing conditions of financial markets. In portfolio selection, performance should be evaluated across multiple scenarios to reflect heterogeneous market regimes and to account for extreme downside risk. Aggregating financial performance into a single matrix can obscure scenario-specific vulnerabilities and lead to an overly optimistic assessment of portfolio resilience.

The central contribution of this work is the extension of the hybrid MCDM methodology from a deterministic to a stochastic framework that is directly applicable to financial portfolio evaluation. This advancement is realized through two key innovations. First, the framework replaces the traditional single static performance matrix with a collection of scenario-dependent matrices, thereby capturing how alternatives perform under heterogeneous and potentially adverse market conditions. Second, the conventional ranking procedure is reformulated into a robust selection criterion grounded in lightly robust optimization [17]. This shifts the objective from merely identifying the portfolio that appears optimal on aggregated historical data to selecting the portfolio that demonstrates

resilience across all scenarios. In particular, the proposed method identifies the alternative that maximizes worst-case performance while ensuring that its outcome in a nominal scenario remains within a prescribed tolerance of the nominal optimum.

To formalize this approach, let $\mathcal{W} = \{w_0, \ldots, w_{n-1}\}$ be a set of candidate portfolio alternatives. For each scenario $s_q \in \{s_0, s_1, \ldots, s_{p-1}\} =: \mathcal{U}$ where $\mathcal{U}$ has a finite numbers p scenarios and each portfolio $w_i \in \mathcal{W}$, let $x_{ij}^{s_q}$ represent the performance of w_i on criterion $j \in \{0, \ldots, m-1\}$. The hybrid hybrid robustness-MCDM framework proceeds as follows:

Step 1: Scenario-based performance matrix construction

Construct a performance matrix $X^{s_q} = [x_{ij}^{s_q}] \in \mathbb{R}^{n\times m}$ for each scenario s_q, where each row corresponds to a portfolio i and each column j to a criterion.

Step 2: Criteria weighting via entropy method

For each scenario s_q, once the performance matrix $X^{s_q} = [x_{ij}^{s_q}]$ is constructed, entropy-based weights are assigned to each criterion j. Since the relative importance and discriminative power of each criterion may vary under different market conditions, the weighting process is carried out separately for each scenario. This ensures that criteria with greater variability and higher information content in a given scenario are appropriately emphasized in the decision-making process.

First, normalize each column:

$$z_{ij}^{s_q} = \frac{x_{ij}^{s_q}}{\sum_{i=0}^{n-1} x_{ij}^{s_q}}$$

Then compute the entropy and degree of diversification:

$$E_j^{s_q} = -\frac{1}{\ln n} \sum_{i=0}^{n-1} z_{ij}^{s_q} \ln(z_{ij}^{s_q}), \quad d_j^{s_q} = 1 - E_j^{s_q}$$

Finally, calculate the normalized entropy-based weights:

$$\omega_j^{s_q} = \frac{d_j^{s_q}}{\sum_{j=0}^{m-1} d_j^{s_q}}$$

Step 3: Vector normalization and ideal solutions (TOPSIS)

To ensure that performance values across different criteria are comparable and dimensionless, the decision matrix is normalized using vector normalization. This step eliminates the influence of differing scales or units among the criteria and prepares the data for distance-based comparison.

Normalize the decision matrix:

$$y_{ij}^{s_q} = \frac{x_{ij}^{s_q}}{\sqrt{\sum_{i=0}^{n-1} (x_{ij}^{s_q})^2}}$$

After normalization, the technique for order preference by similarity to ideal solution (TOPSIS) defines benchmark reference points: the positive ideal solution (PIS), which represents the best achievable performance for each criterion, and the negative ideal solution (NIS), which corresponds to the worst.

Determine the ideal and anti-ideal values:

$$PIS_j^{s_q} = \max_i y_{ij}^{s_q}, \quad NIS_j^{s_q} = \min_i y_{ij}^{s_q}$$

Step 4: Grey relational coefficients

After identifying the ideal benchmarks, the next step assesses how closely each alternative approximates these reference points. To do this, we compute deviations from the positive and negative ideal solutions for each criterion:

$$\Delta_{ij}^{+,s_q} = \left| y_{ij}^{s_q} - PIS_j^{s_q} \right|, \quad \Delta_{ij}^{-,s_q} = \left| y_{ij}^{s_q} - NIS_j^{s_q} \right|$$

These deviations quantify how far each normalized value lies from the best and worst possible performances, respectively.

To transform these deviations into similarity measures, we apply GRA, which evaluates the relative closeness between each alternative and the ideal solutions using grey relational coefficients. With a distinguishing coefficient $\xi \in (0, 1]$, the coefficients are computed as:

$$G_{ij}^{s_q} = \frac{m^{+,s_q} + \xi M^{+,s_q}}{\Delta_{ij}^{+,s_q} + \xi M^{+,s_q}}, \quad V_{ij}^{s_q} = \frac{m^{-,s_q} + \xi M^{-,s_q}}{\Delta_{ij}^{-,s_q} + \xi M^{-,s_q}}$$

where:

$$M^{+,s_q} = \max_j \max_i \Delta_{ij}^{+,s_q}, \quad m^{+,s_q} = \min_j \min_i \Delta_{ij}^{+,s_q}$$

$$M^{-,s_q} = \max_j \max_i \Delta_{ij}^{-,s_q}, \quad m^{-,s_q} = \min_j \min_i \Delta_{ij}^{-,s_q}$$

Here, $G_{ij}^{s_q}$ reflects the closeness of alternative i to the positive ideal, while $V_{ij}^{s_q}$ reflects proximity to the negative ideal. The use of both coefficients enables a balanced assessment of an alternative's performance, considering both its strengths and weaknesses relative to scenario-specific benchmarks.

Step 5: Scenario-wise robust priority index (EDAS-inspired)

To assess the relative performance of each portfolio alternative under scenario s_q, we apply a modified EDAS strategy that leverages the previously computed grey relational coefficients. This step evaluates how much each alternative deviates from the average performance, distinguishing between favorable and unfavorable deviations, and balances these effects to produce a scenario-specific score.

For each scenario s_q, the following steps are performed for each alternative w_i:

(1) Compute average reference levels:

These serve as scenario-specific benchmarks representing the average similarity to the ideal and anti-ideal solutions across all alternatives.

$$AV_{j,\text{best}}^{s_q} = \frac{1}{n} \sum_i G_{ij}^{s_q}, \quad AV_{j,\text{worst}}^{s_q} = \frac{1}{n} \sum_i V_{ij}^{s_q}$$

(2) Calculate positive and negative deviation analysis:

These values quantify how much an alternative improves upon (positive deviation) or falls below (negative deviation) the average reference level, normalized by that reference level. The use of $\max(0, \cdot)$ ensures that only meaningful deviations are considered.

$$PDA_{ij,\text{best}}^{s_q} = \max\left(0, \frac{G_{ij}^{s_q} - AV_{j,\text{best}}^{s_q}}{AV_{j,\text{best}}^{s_q}}\right),$$
$$NDA_{ij,\text{best}}^{s_q} = \max\left(0, \frac{AV_{j,\text{best}}^{s_q} - G_{ij}^{s_q}}{AV_{j,\text{best}}^{s_q}}\right),$$
$$PDA_{ij,\text{worst}}^{s_q} = \max\left(0, \frac{V_{ij}^{s_q} - AV_{j,\text{worst}}^{s_q}}{AV_{j,\text{worst}}^{s_q}}\right),$$
$$NDA_{ij,\text{worst}}^{s_q} = \max\left(0, \frac{AV_{j,\text{worst}}^{s_q} - V_{ij}^{s_q}}{AV_{j,\text{worst}}^{s_q}}\right)$$

(3) Weight the deviations using entropy-based weights:

The entropy-derived weights $\omega_j^{s_q}$ are used to combine the deviations across all criteria, reflecting the importance of each criterion under scenario s_q.

$$SP_{i,\text{best}}^{s_q} = \sum_j \omega_j^{s_q} \cdot PDA_{ij,\text{best}}^{s_q}, \quad SN_{i,\text{best}}^{s_q} = \sum_j \omega_j^{s_q} \cdot NDA_{ij,\text{best}}^{s_q},$$

$$SP_{i,\text{worst}}^{s_q} = \sum_j \omega_j^{s_q} \cdot PDA_{ij,\text{worst}}^{s_q}, \quad SN_{i,\text{worst}}^{s_q} = \sum_j \omega_j^{s_q} \cdot NDA_{ij,\text{worst}}^{s_q}$$

(4) Normalize the deviation scores:

Each weighted score is normalized relative to the best-performing alternative, ensuring that the resulting metrics fall in the range $[0, 1]$, where higher values indicate a higher priority in the decision-making process.

$$NSP_{i,\text{best}}^{s_q} = \frac{SP_{i,\text{best}}^{s_q}}{\max_k SP_{k,\text{best}}^{s_q}}, \quad NSN_{i,\text{best}}^{s_q} = 1 - \frac{SN_{i,\text{best}}^{s_q}}{\max_k SN_{k,\text{best}}^{s_q}}$$
$$NSP_{i,\text{worst}}^{s_q} = \frac{SP_{i,\text{worst}}^{s_q}}{\max_k SP_{k,\text{worst}}^{s_q}}, \quad NSN_{i,\text{worst}}^{s_q} = 1 - \frac{SN_{i,\text{worst}}^{s_q}}{\max_k SN_{k,\text{worst}}^{s_q}}$$

(5) Aggregate normalized profiles:

The best and worst profiles are averaged to produce two scenario-specific appraisal scores for each alternative.

$$AS_{i,\text{best}}^{s_q} = \frac{NSP_{i,\text{best}}^{s_q} + NSN_{i,\text{best}}^{s_q}}{2}, \quad AS_{i,\text{worst}}^{s_q} = \frac{NSP_{i,\text{worst}}^{s_q} + NSN_{i,\text{worst}}^{s_q}}{2}$$

(6) Compute the final scenario-wise appraisal score:

The final score for each alternative under scenario s_q is derived by combining its similarity to the best and dissimilarity from the worst. This balanced formulation rewards both strong performance and resilience.

$$AS_i^{s_q} = \frac{AS_{i,\text{best}}^{s_q} + (1 - AS_{i,\text{worst}}^{s_q})}{2}$$

Step 6: Scenario aggregation and robust portfolio selection

To integrate scenario-wise performance with robustness, we apply the lightly robust optimization principle to the appraisal scores $AS_i^{s_q}$ obtained in Step 5.

Let $\hat{s} \in \mathcal{U}$ denote the nominal scenario. Following the definition of lightly robust optimality in Definition 2.1, we obtain the final robust priority index by solving:

$$AS_{i^*}^{\epsilon} := \max_{w_i \in \mathcal{W}} \left[\min_{s_q \in \mathcal{U}} AS_i^{s_q} \right], \tag{2.4}$$

$$\text{subject to} \quad AS_i^{\hat{s}} \geq \max_j AS_j^{\hat{s}} - \epsilon. \tag{2.5}$$

This formulation identifies the portfolio alternative that achieves the highest worst-case performance across all scenarios, subject to the constraint that its performance under the nominal scenario remains within a tolerance ϵ of the nominal best. It therefore balances robustness under uncertainty with near-optimality under normal market conditions.

Example 2.1. *To clarify the practical application of the lightly robust optimization principle (Step 6) and the role of the tolerance parameter ϵ, we present a simplified decision problem. Assume that after the MCDM evaluation (Steps 1–5), three candidate portfolios have been assessed: (A) Aggressive Growth, (B) Balanced, and (C) All-Weather. The goal is to select the most suitable portfolio by balancing its performance in a nominal scenario (representing normal market conditions) with its resilience in adverse scenarios.*

The key performance metrics required for this step are the appraisal score for each portfolio under the nominal scenario ($AS_i^{\hat{s}}$) and its worst-case score across all considered scenarios ($\min_{s_q} AS_i^{s_q}$). These are summarized in Table 1.

TABLE 1. Appraisal scores for candidate portfolios

Portfolio alternative	***Description***	$AS_i^{\hat{s}}$	$\min_{s_q} AS_i^{s_q}$
A	*Aggressive Growth*	*0.95*	*0.20*
B	*Balanced*	*0.93*	*0.45*
C	*All-Weather*	*0.91*	*0.60*

From Table 1, Portfolio A is an optimal solution in nominal scenario $\hat{s}$ represented as the fair-weather champion, achieving the highest score under nominal conditions ($\max_j AS_j^{\hat{s}} = 0.95$). In contrast, Portfolio C is the most robust, evidenced by its superior worst-case score. The final selection is determined by the decision-maker's specified value for ϵ, which quantifies their tolerance for sacrificing nominal performance to gain robustness.

We now evaluate the selection for different values of ϵ.

Case 1: $\epsilon = 0.01$ (Low tolerance for sub-optimality). *The constraint from Eq.* (2.5) *requires the selected portfolio's nominal score to be at least $0.95 - 0.01 = 0.94$.*

- *Feasible Set: Only Portfolio A (with a nominal score of 0.95) satisfies this stringent condition.*
- *Selection: The framework selects Portfolio A. The decision-maker's preference heavily favors nominal performance, using robustness only as a potential tie-breaker.*

Case 2: $\epsilon = 0.03$ (Moderate tolerance). *The constraint requires the nominal score to be at least $0.95 - 0.03 = 0.92$.*

- *Feasible Set: Portfolios A (0.95) and B (0.93) are now feasible, while Portfolio C (0.91) remains excluded.*
- *Selection: The framework applies the maximin objective from Eq.* (2.4) *to this reduced set. It compares the worst-case scores of the feasible alternatives: $\max\{0.20, 0.45\} = 0.45$. This corresponds to Portfolio B. The framework selects Portfolio B.*

Case 3: $\epsilon = 0.05$ (Higher tolerance for gaining robustness). *The constraint requires the nominal score to be at least* $0.95 - 0.05 = 0.90$.

- *Feasible Set: All three portfolios—A (0.95), B (0.93), and C (0.91)—are now considered feasible.*
- *Selection: The framework applies the maximin objective to the full set, seeking the portfolio with the highest worst-case score:* $\max\{0.20, 0.45, 0.60\} = 0.60$. *This corresponds to Portfolio C. The framework selects Portfolio C.*

This example demonstrates that ϵ functions as a tuning parameter for risk aversion. A small ϵ confines the choice to portfolios that are near-optimal under normal conditions. As ϵ increases, the decision-maker signals a greater willingness to trade nominal performance for superior resilience against adverse scenarios. The choice of ϵ thus allows for an explicit and quantifiable definition of the preferred balance on the trade-off frontier between optimality and robustness.

While the preceding example illustrates the inherent trade-off between optimality and robustness, a formal framework is required to quantify this balance. In particular, we need a systematic method to evaluate the performance loss incurred for a given improvement in resilience.

2.3. **Balancing robustness and nominal performance in ranking.** As outlined in Step 6 of the proposed framework, the representative ranking score $AS_{i^*}^{\epsilon}$ is determined with respect to a relaxation threshold parameter ϵ, which specifies the allowable deviation from the nominally optimal appraisal score. This relaxation permits the inclusion of additional portfolios w_k whose nominal appraisal score $AS_k^{\hat{s}}$ remains within a prescribed margin of the best-performing solution under the nominal scenario $\hat{s}$. To evaluate the trade-off between robustness and nominal ranking quality as a function of ϵ, two complementary metrics are introduced: the *price of ranking for robustness*, which quantifies the nominal performance sacrificed, and the *gain of ranking for robustness*, which measures the improvement in robustness achieved.

For each relaxation threshold $\epsilon \geq 0$ considered in Step 6, we denote the feasible set, as defined in (2.5), by

$$\mathcal{W}_{\epsilon} := \left\{ k \in I \;\middle|\; AS_k^{\hat{s}} \geq \max_{j \in I} AS_j^{\hat{s}} - \epsilon \right\}. \tag{2.6}$$

Subsequently, we compute *price of ranking for robustness* as

$$\text{Price}(AS_{i^*}^{\epsilon}, \hat{s}) = \max_{k \in \mathcal{W}_{\epsilon}} \left\{ AS_{\text{nom}}^{\hat{s}} - AS_k^{\hat{s}} \right\}, \tag{2.7}$$

where $AS_{i^*}^{\epsilon}$ represents the final ranking score obtained under relaxation threshold ϵ and $AS_{\text{nom}}^{\hat{s}}$ denotes the appraisal score of the optimal portfolio in the nominal scenario $\hat{s}$. This metric measures the maximum loss in nominal ranking quality that must be accepted to obtain improved in robustness. Conversely, the *gain of ranking for robustness* is computed as

$$\text{Gain}(AS_{i^*}^{\epsilon}, \hat{s}) = \max_{k \in \mathcal{W}_{\epsilon}} \left\{ \min_{s_q \in \mathcal{U}} AS_k^{s_q} - \min_{s_q \in \mathcal{U}} AS_{\text{nom}}^{s_q} \right\}, \tag{2.8}$$

Here, $AS_{\text{nom}}^{s_q}$ denotes the appraisal score of the nominally optimal portfolio in scenario s_q. This metric captures the improvement in worst-case ranking quality resulting from relaxing the nominal optimality requirement.

3. Theoretical Analysis and Sensitivity of the Robustness–MCDM Framework

This section develops the theoretical underpinnings of the proposed robustness–MCDM framework. We investigate how small perturbations in the underlying performance data affect the appraisal scores and relative rankings. In particular, we establish properties of monotonicity, define criterion-wise and system-level perturbation measures, and introduce sensitivity indices that capture the minimal adjustments required for ranking changes.

3.1. **Theoretical properties of perturbation effects.**

Theorem 3.1 (Element-wise monotonicity)**.** *Let $x_{ij}^{s_q}$ denote the performance of portfolio w_i on criterion j under scenario $s_q \in \mathcal{U}$. Define two performance vectors for w_i under this scenario as*

$$\mathbf{x}_i^{s_q} = [x_{i0}^{s_q}, \ldots, x_{ij}^{s_q}, \ldots, x_{i(m-1)}^{s_q}],$$

$$\mathbf{x'}_i^{s_q} = [x_{i0}^{s_q}, \ldots, x_{ij}^{s_q} + \epsilon, \ldots, x_{i(m-1)}^{s_q}],$$

where $\epsilon \in (0, x_{ij}^{s_q})$ is a positive increment. Let $\mathrm{AS}_i^{s_q}$ and $\mathrm{AS'}_i^{s_q}$ denote the appraisal scores computed using $\mathbf{x}_i^{s_q}$ and $\mathbf{x'}_i^{s_q}$, respectively.

Assume that all other inputs and weights are held constant and that the following conditions are satisfied:

(i) The term Δ_{ij}^{+,s_q} corresponding to $x_{ij}^{s_q}$ is not an extremum:

$$\Delta_{ij}^{+,s_q} \neq \max_j \max_i \Delta_{ij}^{+,s_q} \quad \textit{and} \quad \Delta_{ij}^{+,s_q} \neq \min_j \min_i \Delta_{ij}^{+,s_q}.$$

(ii) The term Δ_{ij}^{-,s_q} corresponding to $x_{ij}^{s_q}$ is not an extremum:

$$\Delta_{ij}^{-,s_q} \neq \max_j \max_i \Delta_{ij}^{-,s_q} \quad \textit{and} \quad \Delta_{ij}^{-,s_q} \neq \min_j \min_i \Delta_{ij}^{-,s_q}.$$

(iii) The average value

$$AV_j^{s_q} = \frac{1}{n} \sum_k G_{kj}^{s_q}$$

remains unchanged when calculated using the perturbed vector $\mathbf{x'}_i^{s_q}$.

(iv) The deviation-based quantities do not attain the maximum values across alternatives, i.e.,

$$SP_i^{s_q} \neq \max_k SP_{k,best}^{s_q}, \quad SP_i^{s_q} \neq \max_k SP_{k,worst}^{s_q}, \quad SN_i^{s_q} \neq \max_k SN_{k,best}^{s_q}, \quad SN_i^{s_q} \neq \max_k SN_{k,worst}^{s_q}.$$

(v) After perturbation by ϵ, conditions (i)–(iv) continue to hold for $\mathbf{x'}_i^{s_q}$.

(vi) Assume that the perturbed and unperturbed similarity and dissimilarity scores do not cross the average, i.e.,

$$\big(G_{ij}^{s_q} - AV_j^{s_q}\big)\big(G_{ij}^{\prime,s_q} - AV_j^{s_q}\big) \geq 0 \quad \textit{and} \quad \big(V_{ij}^{s_q} - AV_j^{s_q}\big)\big(V_{ij}^{\prime,s_q} - AV_j^{s_q}\big) \geq 0.$$

Then the appraisal score is non-decreasing:

$$\mathrm{AS'}_i^{s_q} \;\geq\; \mathrm{AS}_i^{s_q}.$$

Proof. We prove that for any portfolio w_i, criterion j, and scenario $s_q \in \mathcal{U}$, if the performance value $x_{ij}^{s_q}$ is increased by an amount $\epsilon \in (0, x_{ij}^{s_q})$ while all other inputs remain fixed, then the corresponding appraisal score $\mathrm{AS}_i^{s_q}$ is non-decreasing. Throughout the proof, notations with a prime ($'$) are derived from $\mathbf{x'}_i^{s_q}$, whereas notations without a prime are derived from $\mathbf{x}_i^{s_q}$.

Step 1. Let $i \in I$ and fix $\epsilon \in (0, x_{ij}^{s_q})$. The normalized value of $x_{ij}^{s_q}$ is

$$y_{ij}^{s_q} = \frac{x_{ij}^{s_q}}{\sqrt{\sum_{k=0}^{n-1}(x_{kj}^{s_q})^2}} = \frac{x_{ij}^{s_q}}{\sqrt{(x_{ij}^{s_q})^2 + A}}, \quad \text{where } A = \sum_{k\neq i}(x_{kj}^{s_q})^2. \tag{3.9}$$

After perturbation, the new normalized value is

$$y'^{s_q}_{ij} = \frac{x_{ij}^{s_q} + \epsilon}{\sqrt{(x_{ij}^{s_q} + \epsilon)^2 + A}}. \tag{3.10}$$

Define $f(x) = \frac{x}{\sqrt{x^2+A}}$. Since

$$\frac{df}{dx} = \frac{A}{(x^2 + A)^{3/2}} > 0 \quad \text{for all } x > 0,$$

f is strictly increasing on $(0, \infty)$. Because $x_{ij}^{s_q} + \epsilon > x_{ij}^{s_q}$, we have

$$y_{ij}^{s_q} = f(x_{ij}^{s_q}) < f(x_{ij}^{s_q} + \epsilon) = y'^{,s_q}_{ij}.$$

Step 2. We now show that Δ_{ij}^{+,s_q} is non-increasing and Δ_{ij}^{-,s_q} is non-decreasing in $x_{ij}^{s_q}$.

Since $y'^{,s_q}_{ij} > y_{ij}^{s_q}$ and $y'^{,s_q}_{kj} < y_{kj}^{s_q}$ for $k \neq i$, two cases arise for the PIS:

Case I: If the PIS is determined by $m \neq i$, then $PIS'_j \leq PIS_j$ while $y'^{,s_q}_{ij} > y_{ij}^{s_q}$, which implies

$$\Delta'^{,+,s_q}_{ij} < \Delta_{ij}^{+,s_q}.$$

Case II: If the PIS is determined by i, then $PIS'_j = y'^{,s_q}_{ij}$, so

$$\Delta'^{,+,s_q}_{ij} = 0 \leq \Delta_{ij}^{+,s_q}.$$

Thus Δ_{ij}^{+,s_q} is non-increasing.

Similarly, for the NIS, we know $NIS'_j \leq NIS_j$ and $y'^{,s_q}_{ij} \geq y_{ij}^{s_q}$. Hence,

$$\Delta'^{,-,s_q}_{ij} = y'^{,s_q}_{ij} - NIS'_j \geq y_{ij}^{s_q} - NIS_j = \Delta_{ij}^{-,s_q}.$$

Therefore, Δ_{ij}^{-,s_q} is non-decreasing.

Step 3. Using Step 2, we analyze the scores

$$G_{ij}^{s_q} = \frac{m^{+,s_q} + \xi M^{+,s_q}}{\Delta_{ij}^{+,s_q} + \xi M^{+,s_q}}, \qquad V_{ij}^{s_q} = \frac{m^{-,s_q} + \xi M^{-,s_q}}{\Delta_{ij}^{-,s_q} + \xi M^{-,s_q}}.$$

Because $G_{ij}^{s_q}$ decreases with Δ_{ij}^{+,s_q} and Δ_{ij}^{+,s_q} is non-increasing, it follows that $G_{ij}^{s_q}$ is non-decreasing in $x_{ij}^{s_q}$. Similarly, since $V_{ij}^{s_q}$ decreases with Δ_{ij}^{-,s_q} and Δ_{ij}^{-,s_q} is non-decreasing, we conclude that $V_{ij}^{s_q}$ is non-increasing.

Step 4. Let $AV_j^{s_q} = \frac{1}{n}\sum_i G_{ij}^{s_q}$ denote the average grey similarity. By assumption (III), $AV_j^{s_q}$ remains unchanged after perturbation. Define

$$PDA_{ij,\text{best}}^{s_q} = \max\left(0, \frac{G_{ij}^{s_q} - AV_j^{s_q}}{AV_j^{s_q}}\right), \qquad NDA_{ij,\text{best}}^{s_q} = \max\left(0, \frac{AV_j^{s_q} - G_{ij}^{s_q}}{AV_j^{s_q}}\right),$$

$$PDA_{ij,\text{worst}}^{s_q} = \max\left(0, \frac{V_{ij}^{s_q} - AV_j^{s_q}}{AV_j^{s_q}}\right), \quad NDA_{ij,\text{worst}}^{s_q} = \max\left(0, \frac{AV_j^{s_q} - V_{ij}^{s_q}}{AV_j^{s_q}}\right).$$

From Step 3, perturbation ensures $G_{ij}^{\prime,s_q} \geq G_{ij}^{s_q}$ and $V_{ij}^{\prime,s_q} \leq V_{ij}^{s_q}$. We now consider four possible cases:

Case I: $G_{ij}^{s_q}, G_{ij}^{\prime,s_q} \leq AV_j^{s_q}$ and $V_{ij}^{s_q}, V_{ij}^{\prime,s_q} \leq AV_j^{s_q}$. Then $PDA_{ij,\text{best}}^{s_q} = 0$, $PDA_{ij,\text{worst}}^{s_q} = 0$, and both $NDA_{ij,\text{best}}^{s_q}, NDA_{ij,\text{worst}}^{s_q} > 0$. After perturbation, $NDA_{ij,\text{best}}^{\prime,s_q} \leq NDA_{ij,\text{best}}^{s_q}$ and $NDA_{ij,\text{worst}}^{\prime,s_q} \geq NDA_{ij,\text{worst}}^{s_q}$, while the PDA terms remain zero. Thus,

$$SP_{i,\text{best}}^{\prime,s_q} = 0, \quad SP_{i,\text{worst}}^{\prime,s_q} = 0, \quad SN_{i,\text{best}}^{\prime,s_q} \leq SN_{i,\text{best}}^{s_q}, \quad SN_{i,\text{worst}}^{\prime,s_q} \geq SN_{i,\text{worst}}^{s_q},$$

which implies

$$AS_{i,\text{best}}^{\prime\, s_q} \geq AS_{i,\text{best}}^{s_q}, \qquad AS_{i,\text{worst}}^{\prime\, s_q} \leq AS_{i,\text{worst}}^{s_q}.$$

Case II: $G_{ij}^{s_q}, G_{ij}^{\prime,s_q} \geq AV_j^{s_q}$ and $V_{ij}^{s_q}, V_{ij}^{\prime,s_q} \geq AV_j^{s_q}$. Then $NDA_{ij,\text{best}}^{s_q} = 0$, $NDA_{ij,\text{worst}}^{s_q} = 0$, and both $PDA_{ij,\text{best}}^{s_q}, PDA_{ij,\text{worst}}^{s_q} > 0$. After perturbation, $PDA_{ij,\text{best}}^{\prime,s_q} \geq PDA_{ij,\text{best}}^{s_q}$ and $PDA_{ij,\text{worst}}^{\prime,s_q} \leq PDA_{ij,\text{worst}}^{s_q}$, while the NDA terms remain zero. Thus,

$$SP_{i,\text{best}}^{\prime,s_q} \geq SP_{i,\text{best}}^{s_q}, \quad SP_{i,\text{worst}}^{\prime,s_q} \leq SP_{i,\text{worst}}^{s_q}, \quad SN_{i,\text{best}}^{\prime,s_q} = 0, \quad SN_{i,\text{worst}}^{\prime,s_q} = 0,$$

which again implies

$$AS_{i,\text{best}}^{\prime\, s_q} \geq AS_{i,\text{best}}^{s_q}, \qquad AS_{i,\text{worst}}^{\prime\, s_q} \leq AS_{i,\text{worst}}^{s_q}.$$

Case III: $G_{ij}^{s_q}, G_{ij}^{\prime,s_q} \geq AV_j^{s_q}$ and $V_{ij}^{s_q}, V_{ij}^{\prime,s_q} \leq AV_j^{s_q}$. Then $PDA_{ij,\text{best}}^{s_q} > 0$, $NDA_{ij,\text{worst}}^{s_q} > 0$, while the other two terms vanish. After perturbation, $PDA_{ij,\text{best}}^{\prime,s_q} \geq PDA_{ij,\text{best}}^{s_q}$ and $NDA_{ij,\text{worst}}^{\prime,s_q} \geq NDA_{ij,\text{worst}}^{s_q}$, with the remaining two terms still zero. Consequently,

$$SP_{i,\text{best}}^{\prime,s_q} \geq SP_{i,\text{best}}^{s_q}, \quad SN_{i,\text{worst}}^{\prime,s_q} \geq SN_{i,\text{worst}}^{s_q}, \quad SP_{i,\text{worst}}^{\prime,s_q} = 0, \quad SN_{i,\text{best}}^{\prime,s_q} = 0,$$

leading again to

$$AS_{i,\text{best}}^{\prime\, s_q} \geq AS_{i,\text{best}}^{s_q}, \qquad AS_{i,\text{worst}}^{\prime\, s_q} \leq AS_{i,\text{worst}}^{s_q}.$$

Case IV: $G_{ij}^{s_q}, G_{ij}^{\prime,s_q} \leq AV_j^{s_q}$ and $V_{ij}^{s_q}, V_{ij}^{\prime,s_q} \geq AV_j^{s_q}$. Then $NDA_{ij,\text{best}}^{s_q} > 0$, $PDA_{ij,\text{worst}}^{s_q} > 0$, while the other two terms vanish. After perturbation, $NDA_{ij,\text{best}}^{\prime,s_q} \leq NDA_{ij,\text{best}}^{s_q}$ and $PDA_{ij,\text{worst}}^{\prime,s_q} \leq PDA_{ij,\text{worst}}^{s_q}$, with the remaining two terms still zero. Therefore,

$$SN_{i,\text{best}}^{\prime,s_q} \leq SN_{i,\text{best}}^{s_q}, \quad SP_{i,\text{worst}}^{\prime,s_q} \leq SP_{i,\text{worst}}^{s_q}, \quad SP_{i,\text{best}}^{\prime,s_q} = 0, \quad SN_{i,\text{worst}}^{\prime,s_q} = 0,$$

which again implies

$$AS_{i,\text{best}}^{\prime\, s_q} \geq AS_{i,\text{best}}^{s_q}, \qquad AS_{i,\text{worst}}^{\prime\, s_q} \leq AS_{i,\text{worst}}^{s_q}.$$

In all four cases, the inequalities hold, so we conclude

$$AS_{i,\text{best}}^{\prime\, s_q} \geq AS_{i,\text{best}}^{s_q}, \qquad AS_{i,\text{worst}}^{\prime\, s_q} \leq AS_{i,\text{worst}}^{s_q}.$$

Therefore from Steps 1–4, we conclude that under a perturbation of $x_{ij}^{s_q}$ by $\epsilon > 0$, the scenario appraisal score satisfies

$$AS_i^{\prime\, s_q} = \frac{AS_{i,\text{best}}^{\prime\, s_q} + (1 - AS_{i,\text{worst}}^{\prime\, s_q})}{2} \;\geq\; \frac{AS_{i,\text{best}}^{s_q} + (1 - AS_{i,\text{worst}}^{s_q})}{2} = AS_i^{s_q}.$$

□

Remark 3.1. *This property implies that improving the value of any single beneficial criterion, while holding all other factors constant, does not decrease the appraisal score of a portfolio under the given scenario.*

Based on Theorem 3.1, perturbing a performance value $x_{ij}^{s_q}$ by a small positive quantity ϵ results in a higher appraisal score $AS_i^{s_q}$. Motivated by this result, an intuitive query arises: if we wish to change the ranking position of $AS_i^{s_q}$ so that it reaches the next higher score in the ranking order, what is the minimum perturbation value that must be applied?

3.2. **Perturbation analysis.** We first introduce notations used to formalize the perturbation procedure in this work.

Let $I = \{0, \dots, n-1\}$ denote the set of portfolio alternatives, $\mathcal{U} = \{0, \dots, p-1\}$ the set of scenarios, and $J = \{0, \dots, m-1\}$ the set of evaluation criteria. Each row in the decision matrix corresponds to a pair $(i, q) \in I \times \mathcal{U}$ and each column to a criterion $j \in J$. Define

$$X \in \mathbb{R}^{(np)\times m}, \qquad X_{(i,q),j} := x_{ij}^{s_q},$$

where $x_{ij}^{s_q}$ is the performance of alternative i under scenario q on criterion j. Let $\omega = (\omega_0, \dots, \omega_{m-1})$ be the vector of criterion weights. According to the proposed robustness-MCDM framework procedure, we define a score-generating vector $\Phi(X;\omega)$ as

$$\Phi(X;\omega) \in \mathbb{R}^{np}, \qquad \big(\Phi(X;\omega)\big)_{(i,q)} := AS_i^{s_q},$$

where $AS_i^{s_q}$ denotes the appraisal score of alternative i in scenario q. Let σ be the permutation of indices (i, q) such that

$$\big(\Phi(X;\omega)\big)_{\sigma(1)} \le \big(\Phi(X;\omega)\big)_{\sigma(2)} \le \cdots \le \big(\Phi(X;\omega)\big)_{\sigma(np)}.$$

Definition 3.2 (Criterion-wise differentiated perturbation and sensitivity)**.** *Fix two consecutive rows in the ranking, $a = \sigma(r) = (i, q)$ and $b = \sigma(r+1) = (k, q')$. For a criterion $j \in J$, let $E_{a,j}$ be the single–entry matrix with $(E_{a,j})_{uv} = 1$ if $(u, v) = (a, j)$ and 0 otherwise. Define the criterion-wise perturbation function*

$$g_{a\to b,j}(\delta) := \big(\Phi(X;\omega)\big)_b - \big(\Phi(X + \delta E_{a,j};\omega)\big)_a. \tag{3.11}$$

The minimal perturbation on criterion j required for row a to catch up with row b is

$$\delta^*_{a\to b,j} := \inf\{\delta \ge 0 : g_{a\to b,j}(\delta) = 0\}. \tag{3.12}$$

We then define the criterion-wise differentiated perturbation as

$$\Delta^{\text{diff}}_{i,\,a\to b,\,j} := \frac{\big(\Phi(X;\omega)\big)_b - \big(\Phi(X;\omega)\big)_a}{\delta^*_{a\to b,j}}, \tag{3.13}$$

which quantifies the change in appraisal score required to bridge the gap to the next rank, per unit of perturbation applied to criterion j. For the top-ranked alternative, where no higher neighbor b exists, the index in (3.13) *is not defined; by convention, we set*

$$\Delta^{\text{diff}}_{top,\,a\to null,\,j} := +\infty, \tag{3.14}$$

to indicate that no finite upward perturbation can further improve its position.

Relying on the concept of criterion-wise differentiated perturbation in Definition 3.2, we define the system differentiated perturbation as follows.

Definition 3.3 (System differentiated perturbation)**.** *Using the criterion-wise differentiated perturbation index* (3.13)*, the system differentiated perturbation is defined as*

$$\Delta^{\text{diff}}_{system} := \max_{i\in I,\, q\in\mathcal{U},\, j\in J} \Delta^{\text{diff}}_{i,\,a\to b,j}, \tag{3.15}$$

that is, the smallest adjustment, across all alternatives, criteria, and scenarios, required for any alternative to advance in the ranking.

Remark 3.2. *Under the conditions of Theorem 3.1, the criterion-wise perturbation $\delta^*_{a\to b,j}$ (Definition 3.2) represents the minimal adjustment of $x_{ij}^{s_q}$ such that*

$$g_{a\to b,j}(\delta) \;:=\; \big(\Phi(X;\omega)\big)_b - \big(\Phi(X+\delta E_{a,j};\omega)\big)_a,$$

and we seek the smallest $\delta \geq 0$ with $g_{a\to b,j}(\delta) = 0$. By Theorem 3.1, $AS_i^{s_q}$ is monotone non-decreasing in $x_{ij}^{s_q}$ (and $\big(\Phi(X;\omega)\big)_{(i,q)}$ is a representation of $AS_i^{s_q}$), which implies that $g_{a\to b,j}$ is monotone in δ. In particular, if $g_{a\to b,j}(0) > 0$ and there exists $\bar{\delta} > 0$ with $g_{a\to b,j}(\bar{\delta}) \leq 0$, then a unique root lies in $[0, \bar{\delta}]$. Therefore, $\delta^*_{a\to b,j}$ can be efficiently computed using standard one-dimensional root-finding methods (e.g., bisection, Brent's method, or toms748) on an appropriate bracketing interval.

4. Experimental Setup

4.1. **Data source, time horizon, and portfolio set definition.** To evaluate the effectiveness of the proposed hybrid robustness–MCDM portfolio framework, we use a dataset of daily log returns for publicly traded equities obtained from Yahoo Finance. The dataset spans $T = 1470$ trading days, covering the period from January 3, 2019, to December 30, 2024. The investment universe consists of $d = 25$ assets drawn from diverse economic sectors, including technology, finance, energy, healthcare, and consumer goods.

The portfolio alternative set is defined as

$$\mathcal{W} = \{w_0, w_1, \ldots, w_9\} \subset \mathbb{R}^{25},$$

where each portfolio $w_i = [w_{(i,0)}, \ldots, w_{(i,24)}]^\top$ is a vector of asset weights drawn from the above universe. All portfolios satisfy the full-investment constraint

$$\sum_{m=0}^{24} w_{(i,m)} = 1, \quad \forall i \in \{0, \ldots, 9\}.$$

4.2. **Return matrix construction and scenario generation.** Let $Y \in \mathbb{R}^{T\times d} = \mathbb{R}^{1470\times 25}$ denote the matrix of daily log returns, where each row corresponds to a trading day and each column to an asset. For scenario-based analysis, the dataset is partitioned into

$$n_{\text{scn}} = \frac{T}{n_{\text{day}}} = 35$$

non-overlapping market scenarios, each comprising $n_{\text{day}} = 42$ consecutive trading days. The choice of n_{day} balances statistical reliability, by providing enough observations for stable covariance estimation, with computational tractability (estimating 2 months).

For each scenario $q \in \{0, \ldots, 34\}$, the corresponding return matrix is

$$Y^{(q)} = Y[q \cdot n_{\text{day}} : (q+1) \cdot n_{\text{day}}, :] \in \mathbb{R}^{42\times 25},$$

where the t_{local}-th row $Y^{(q)}_{t_{\text{local}},:} \in \mathbb{R}^{1\times 25}$ contains the returns of all assets on the t_{local}-th day of scenario q. The mapping between local and global day indices is

$$t_{\text{global}} = q \cdot n_{\text{day}} + t_{\text{local}}, \quad t_{\text{local}} \in \{0, \ldots, n_{\text{day}} - 1\}, \quad t_{\text{global}} \in \{0, \ldots, T-1\}.$$

For example, for $q = 3$ we have

$$Y^{(3)} = Y[126 : 168, :],$$

corresponding to $t_{\text{global}} = 126$ through 167 in the original dataset.

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https://finance.yahoo.com

4.3. **Scenario-wise portfolio returns.** For each portfolio w_i and scenario s_q, the daily return on global day t_{global} is

$$r_{t_{\text{global}}}^{(i,q)} = w_i \, Y_{t_{\text{local}},:}^{(q)},$$

with t_{local} and t_{global} related as above. The scenario-wise expected return and standard deviation are given by

$$\mathbb{E}[r^{(i,q)}] = \frac{1}{n_{\text{day}}} \sum_{t_{\text{local}}=0}^{n_{\text{day}}-1} r_{t_{\text{global}}}^{(i,q)}, \quad \sigma^{(i,q)} = \sqrt{\frac{1}{n_{\text{day}}-1} \sum_{t_{\text{local}}=0}^{n_{\text{day}}-1} \left(r_{t_{\text{global}}}^{(i,q)} - \mathbb{E}[r^{(i,q)}] \right)^2}.$$

4.4. **Performance criteria definition.** These scenario-wise statistics serve as the basis for computing the Sharpe ratio with respect to six risk-adjusted performance metrics. For each portfolio w_i and scenario s_q, the performance of w_i on criterion j is denoted by $x_{ij}^{s_q}$:

- For criterion $j = 0$, the Sharpe ratio is defined according to the Mean–Standard Deviation Ratio (MSV):

$$x_{i0}^{s_q} = \frac{\mathbb{E}[r^{(i,q)}]}{\sigma^{(i,q)}}.$$

- For criterion $j = 1$, the Sharpe ratio is defined according the semi-lower partial moment (SLPM) of order p:

$$x_{i1}^{s_q} = \frac{\mathbb{E}\left[r^{(i,q)}\right]}{\left(\mathbb{E}\left[\left(\max\{0, \tau - r_{t_{\text{global}}}^{(i,q)}\} \right)^p \right] \right)^{1/p}},$$

  where τ is a shortfall threshold and $p = 2$.
- For criterion $j = 2$, the Sharpe ratio is defined according to the Conditional Value-at-Risk (CVaR):

$$x_{i2}^{s_q} = \frac{\mathbb{E}[r^{(i,q)}]}{\mathbb{E}\left[r_{t_{\text{global}}}^{(i,q)} \;\middle|\; r_{t_{\text{global}}}^{(i,q)} \le \text{VaR}_\alpha^{(i,q)} \right]}.$$

- For criterion $j = 3$, the Sharpe ratio is defined according to the Conditional Value-at-Risk Gain (CVRG):

$$x_{i3}^{s_q} = \frac{\mathbb{E}[r^{(i,q)}]}{\text{CVaR}_\alpha^{(i,q)}}.$$

- For criterion $j = 4$, the Sharpe ratio is defined according to the Conditional Drawdown-at-Risk (CDaR):

$$x_{i4}^{s_q} = \frac{\mathbb{E}[r^{(i,q)}]}{\mathbb{E}\left[D_t^{(i,q)} \mid D_t^{(i,q)} \ge \text{DaR}_\alpha^{(i,q)} \right]}.$$

- For criterion $j = 5$, the Sharpe ratio is defined according to the Relative Conditional Drawdown-at-Risk (CDaR_Rel):

$$x_{i5}^{s_q} = \frac{\mathbb{E}[r^{(i,q)}]}{\text{CDaR}_\alpha^{(i,q)}}.$$

Together, these six criteria cover the main risk dimensions in portfolio evaluation: volatility (MSV), downside deviation below a shortfall threshold (SLPM), tail losses (CVaR), reward-to-tail efficiency (CVRG), and path-dependent drawdown risks (CDaR and CDaR_Rel). As a result, the framework captures the key risk dimensions needed to support portfolio selection under uncertainty using Sharpe ratio across multiple risk measures.

5. NUMERICAL EXAMPLES

5.1. **Nominal scenario identification.** Based on the experimental setup in Section 4 and the proposed framework from Steps 1–5, we obtain $AS_i^{s_q}$ for each scenario s_q, $q \in \{0, \ldots, 34\}$, and each portfolio alternative w_i, $i \in \{0, \ldots, 9\}$. The appraisal score matrix is given by

$$A = \begin{bmatrix} AS_0^{s_0} & AS_1^{s_0} & \cdots & AS_9^{s_0} \\ AS_0^{s_1} & AS_1^{s_1} & \cdots & AS_9^{s_1} \\ \vdots & \vdots & \ddots & \vdots \\ AS_0^{s_{34}} & AS_1^{s_{34}} & \cdots & AS_9^{s_{34}} \end{bmatrix} \in \mathbb{R}^{35\times 10}, \tag{5.16}$$

where the q-th row corresponds to scenario s_q, and the i-th column corresponds to portfolio w_i.

To perform Step 6, we identify the *nominal scenario*, defined as the scenario whose appraisal score profile $AS_i^{s_q}$ across all portfolios is closest to the average portfolio behavior across all scenarios. Formally, the nominal scenario is given by

$$\hat{s} \;=\; \arg\min_{s_q \in \mathcal{U}} d(s_q), \tag{5.17}$$

where the Euclidean distance measure is defined as

$$d(s_q) \;=\; \left\| AS^{s_q} - \bar{AS} \right\|_2, \quad q \in \{0, \ldots, 34\}. \tag{5.18}$$

Here, $AS^{s_q} = [AS_0^{s_q}, AS_1^{s_q}, \ldots, AS_9^{s_q}]$ denotes the score vector for scenario s_q, and $\bar{AS} = [\bar{AS}_0, \bar{AS}_1, \ldots, \bar{AS}_9]$ is the average score vector, computed as

$$\bar{AS}_i \;=\; \frac{1}{35} \sum_{q=0}^{34} AS_i^{s_q}, \quad i \in \{0, 1, \ldots, 9\}. \tag{5.19}$$

Applying this procedure, the nominal scenario was identified as $\hat{s} := s_2$, representing the scenario whose appraisal score profile is closest to the cross-scenario average. Under this nominal scenario, portfolio w_8 achieved the highest appraisal score of 0.45771.

5.2. **Nominal score relaxation and solution discussion.** Table 2 presents the stepwise construction of the lightly robust solution sets under increasing relaxation thresholds ϵ_m. Starting from the nominally optimal portfolio w_8 at $\epsilon_0 = 0.0000$, the feasible set $\mathcal{W}_{\epsilon_m}$ is gradually expanded by including additional portfolios whose nominal scores are within ϵ_m of the best nominal score $AS_8^{s_2}$. At each step, the portfolio with the highest minimum appraisal score across all scenarios is selected. The results indicate that w_8 remains optimal up to $\epsilon_1 = 0.0084$, after which w_1, and subsequently w_3 and w_4, become optimal as the threshold increases. These transitions reflect the trade-off between preserving the nominal scenario score and improving robustness.

Table 2 also reports the gain and the price of ranking for robustness associated with each threshold ϵ_m, as defined in Equations (2.7)–(2.8). When ϵ_m increases from 0.0000 to 0.2021, the gain rises from 0.0000 to 0.1055, indicating improved robustness. Note that the gain is not strictly increasing with the threshold. For example, between ϵ_3, ϵ_4, and ϵ_5, the gain remains constant while the price continues to rise. This shows that sacrificing the nominal score does not always produce additional robustness gains. Therefore, these results should be interpreted by weighing the loss in the nominal scenario score against the improvement in the worst-case scenario score.

Accordingly, the trade-off ratio between gain and price at each threshold is reported in the last column of Table 2. The trade-off ratio reaches its maximum value of 5.0013 at $\epsilon_3 = 0.0128$, where the selected portfolio is w_3. This provides a practical guideline for choosing a suitable threshold when determining the final robust priority index in Step 6.

To evaluate the proposed hybrid robustness–MCDM portfolio framework, we also solve the experimental setup in Section 4 using three standard solution concepts to parallel Step 6: a deterministic approach, a minimax robust approach [2], and a stochastic programming approach [3]. In this context, the deterministic approach selects a portfolio that is optimal for a single assumed scenario and therefore does not hedge against scenario variation. The minimax robust approach selects a portfolio by prioritizing protection against the most adverse scenario, that is, by maximizing the worst-case appraisal score over $\mathcal{U}$. The stochastic programming approach selects a portfolio by prioritizing average performance across scenarios, that is, by maximizing the expected appraisal score over $\mathcal{U}$.

Accordingly, the deterministic portfolio w_1 is obtained by ignoring scenario uncertainty and solving the model for a single scenario, that is, $|\mathcal{U}| = 1$. The minimax robust portfolio w_4 is obtained by maximizing the worst-case appraisal score over $\mathcal{U}$, that is,

$$\max_{w_i \in \mathcal{W}} \left[\min_{s_q \in \mathcal{U}} AS_i^{s_q} \right], \tag{5.20}$$

$$\text{subject to} \quad w_i \in \mathcal{W}. \tag{5.21}$$

The stochastic portfolio w_8 is obtained by maximizing the expected appraisal score across scenarios in $\mathcal{U}$, that is,

$$\max_{w_i \in \mathcal{W}} \left[\frac{1}{|\mathcal{U}|} \sum_{s_q \in \mathcal{U}} AS_i^{s_q} \right], \tag{5.22}$$

$$\text{subject to} \quad w_i \in \mathcal{W}. \tag{5.23}$$

TABLE 2. Stepwise expansion of feasible sets $\mathcal{W}_{\epsilon_m}$ and the corresponding lightly robust selections under increasing relaxation thresholds ϵ_m. The table also reports the gain, price, and trade-off ratio at each threshold.

Lightly robust solution concept					
Threshold ϵ_m	**Feasible set $\mathcal{W}_{\epsilon_m}$**	**Selected**	**Gain**	**Price**	**Trade-off**
$\epsilon_0 = 0.0000$	$\{w_8\}$	w_8	0.0000	0.0000	0.0000
$\epsilon_1 = 0.0084$	$\{w_8, w_0\}$	w_8	0.0000	0.0084	0.0000
$\epsilon_2 = 0.0123$	$\{w_8, w_0, w_1\}$	w_1	0.0249	0.0123	2.0249
$\epsilon_3 = 0.0128$	$\{w_8, w_0, w_1, w_3\}$	w_3	0.0642	0.0128	5.0013
$\epsilon_4 = 0.0512$	$\{w_8, w_0, w_1, w_3, w_5\}$	w_3	0.0642	0.0512	1.2543
$\epsilon_5 = 0.0571$	$\{w_8, w_0, w_1, w_3, w_5, w_9\}$	w_3	0.0642	0.0571	1.1233
$\epsilon_6 = 0.0900$	$\{w_8, w_0, w_1, w_3, w_5, w_9, w_4\}$	w_4	0.1055	0.0900	1.1730
$\epsilon_7 = 0.1236$	$\{w_8, w_0, w_1, w_3, w_5, w_9, w_4, w_7\}$	w_4	0.1055	0.1236	0.8540
$\epsilon_8 = 0.1738$	$\{w_8, w_0, w_1, w_3, w_5, w_9, w_4, w_7, w_6\}$	w_4	0.1055	0.1738	0.6072
$\epsilon_9 = 0.2021$	$\{w_8, w_0, w_1, w_3, w_5, w_9, w_4, w_7, w_6, w_2\}$	w_4	0.1055	0.2021	0.5221

5.3. **Analysis of the appraisal scores of the selected portfolios.** Figure 1 illustrates the distribution of appraisal scores across all scenarios for the selected portfolios w_1, w_3, w_4, and w_8. Under the nominal scenario, the appraisal scores are highest for w_8, followed by w_1, w_3, and w_4. This ordering is consistent with the portfolios selected along the relaxation path as the nominal-score threshold increases. Note that the median appraisal score does not necessarily follow the same ordering as the nominal scenario score and is therefore not directly tied to the relaxation threshold. Along the selected sequence w_8, w_1, w_3, and w_4, the worst-case appraisal scores improve as more relaxation is allowed, as reflected by the lower tails of the distributions. This pattern explains why the proposed ranking method can move from the nominally best portfolio toward alternatives with improved worst-case

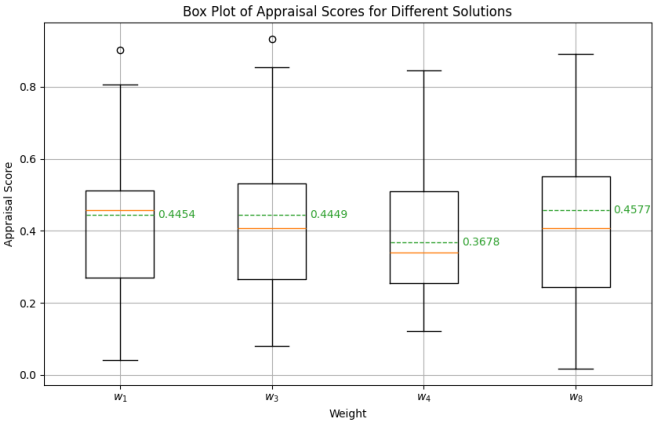


FIGURE 1. Box plot of appraisal scores for the selected portfolio alternatives w_1, w_3, w_4, and w_8. Outliers are shown as circles, the box represents the interquartile range (IQR), and the whiskers summarize the remaining spread. The orange line indicates the median, and the green dashed line indicates the appraisal scores $AS_1^{s_2}, AS_3^{s_2}, AS_4^{s_2}$, and $AS_8^{s_2}$ evaluated under the nominal scenario s_2. The lower tail summarizes each portfolio's worst-case appraisal score.

behavior as ϵ_m increases. In particular, w_3, which is selected at the maximum gain–price trade-off ratio, provides a balanced compromise: its nominal scenario appraisal score is slightly lower than those of w_8 and w_1 but higher than that of w_4, while its worst-case appraisal score improves upon w_8 and w_1 but remains below that of w_4.

5.4. **Analysis of the Sharpe ratios under the risk-adjusted performance criteria.** Figure 2 illustrates the scenario-wise distributions of the Sharpe ratios for the selected portfolios w_1, w_3, w_4, and w_8 under six downside-risk measures. Under the nominal scenario s_2, the portfolio ordering implied by the Sharpe ratio values is consistent with the appraisal-score ordering for MSV (Figure 2a), SLPM (Figure 2b), CDaR (Figure 2e), and CDaR_Rel (Figure 2f). In contrast, for CVaR (Figure 2c) and CVRG (Figure 2d), the nominal ordering differs because the Sharpe ratio value of w_3 exceeds that of w_1. The worst-case behavior, reflected by the lower tails of the box plots, is broadly consistent with the appraisal-score ordering for MSV, SLPM, CVRG, CDaR, and CDaR_Rel, but not for CVaR. Overall, these findings support the use of multiple risk measures in the proposed MCDM step: combining multiple Sharpe ratios reduces reliance on any single risk definition and yields a portfolio ranking that is more stable across different downside-risk perspectives.

5.5. **Statistical performance analysis of returns under different robustness criteria.** Table 3 reports the mean returns, worst-case returns, standard deviation returns, and median returns, computed over 35 non-overlapping periods of 42-day accumulated returns.

The deterministic approach selects portfolio w_1, which achieves the highest mean return at 0.0266 among the listed portfolios. It also exhibits the lowest volatility at 0.0660 and a relatively contained worst-case return at -0.1105.

In contrast, the minimax robust approach selects portfolio w_4. As shown in Table 3, w_4 attains the lowest mean return at 0.0175 and the most adverse worst-case return at -0.1912 among the selected portfolios.

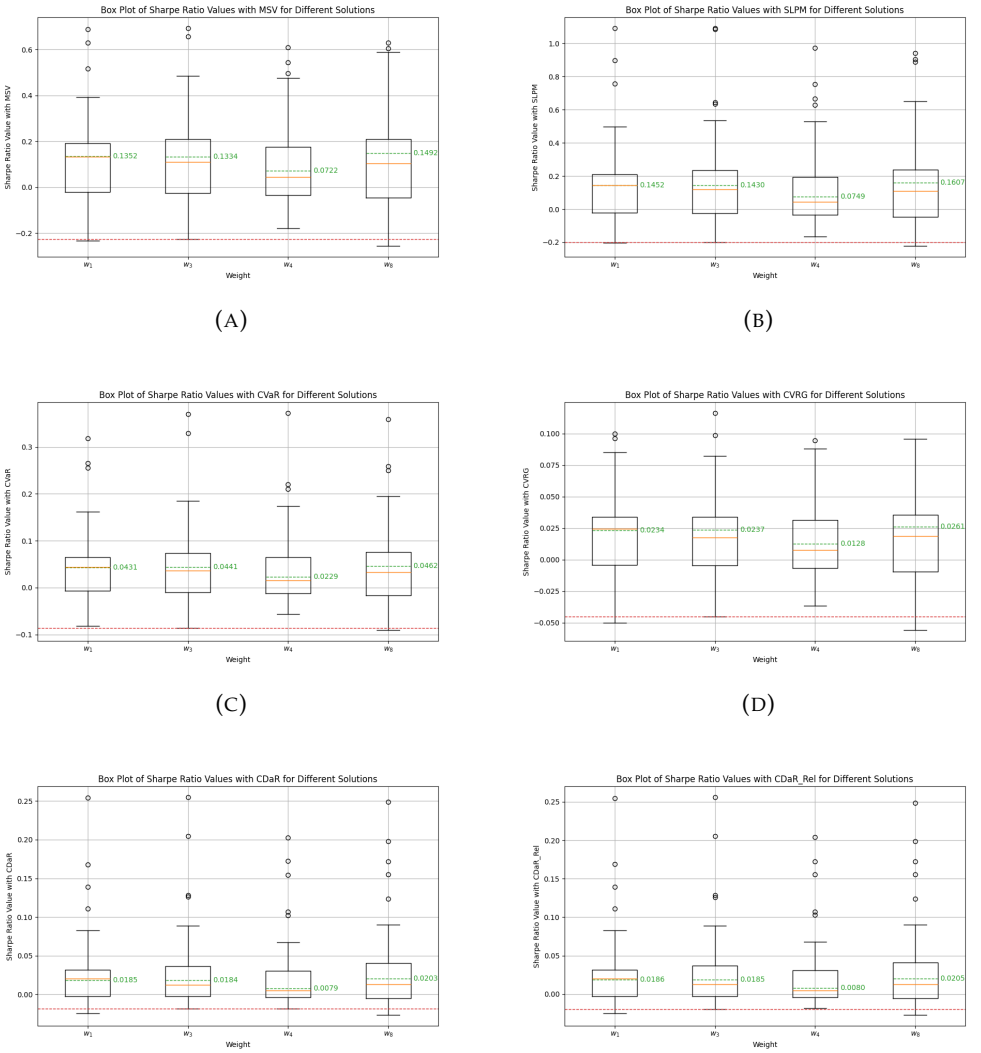


FIGURE 2. Box plots of Sharpe ratio values computed across all scenarios for the selected portfolios w_1, w_3, w_4, and w_8 under six risk measures. Outliers are shown as circles, the box represents the interquartile range (IQR), and the whiskers summarize the remaining spread. The orange line indicates the median, and the green dashed line indicates the Sharpe ratio value under the nominal scenario s_2. The lower tail summarizes each portfolio's worst-case Sharpe ratio value, and the red dashed line highlights the Sharpe ratio value of w_3 for comparison with the other selected portfolios. Subfigures: (A) MSV ($j = 0$), (B) SLPM ($j = 1$), (C) CVaR ($j = 2$), (D) CVRG ($j = 3$), (E) CDaR ($j = 4$), and (F) CDaR_Rel ($j = 5$).

Under the stochastic programming approach, portfolio w_8 is selected. Portfolio w_8 achieves a mean return at 0.0263, close to that of w_1, but it exhibits higher volatility at 0.0732 and a slightly worse worst-case return at -0.1175 than w_1. This outcome is consistent with stochastic selection emphasizing average performance under uncertainty without explicitly protecting against extreme adverse scenarios.

The proposed lightly robust framework bridges the gap between nominal and worst-case performance. As detailed in Table 2, relaxing the nominal score threshold yields a sequence of selected portfolios w_8, w_1, w_3, and w_4 as ϵ_m increases. Among these, w_3 attains a mean return of 0.0262 (close to w_1), moderate volatility of 0.0697, and a worst-case return of -0.1429.

TABLE 3. Statistical performance of selected portfolios, calculated over 35 non-overlapping periods of 42-day accumulated returns.

Type	Portfolio	Mean return	Worst return	Std. Dev.	Median
Robustness-TOPSIS framework	w_3	0.0262	-0.1429	0.0697	0.0266
Deterministic	w_1	0.0266	-0.1105	0.0660	0.0326
Minimax Robust	w_4	0.0175	-0.1912	0.0719	0.0122
Stochastic	w_8	0.0263	-0.1175	0.0732	0.0326

In summary, Table 3 shows that the portfolio w_3 selected by the proposed method does not achieve the strongest return-based statistics. In particular, its mean return is slightly lower than those of the deterministic portfolio w_1 and the stochastic portfolio w_8, and its worst-case return is also less favorable than theirs. These results indicate that w_3 is not the best option when judged by returns alone. However, this does not mean that w_3 is a poor choice, because portfolio selection must also account for other considerations, especially risk. Moreover, the results in Section 5.4 confirm that w_3 provides a suitable balance between return and risk, as reflected in the Sharpe ratio values, under both the nominal scenario and adverse scenarios.

5.6. **Analysis of periodic cumulative returns.** Figure 3 presents the box plots of periodic cumulative returns for four selected portfolios w_1, w_3, w_4, and w_8. In this study, the periodic cumulative return refers to the total compounded return achieved by a portfolio over a fixed evaluation horizon of $n_{\text{day}} = 42$ trading days. This is calculated for each of the 35 non-overlapping scenarios as:

$$R_p^{\text{accum.return}} = \left(\prod_{t=0}^{n_{\text{day}}-1} (1 + r_t) \right) - 1, \tag{5.24}$$

where r_t is the portfolio's daily return at day t within a given period p. Calculating this value for every scenario produces a distribution of 35 returns for each portfolio.

5.6.1. *Distributional analysis of portfolio returns.* Figure 3 shows the distributions of periodic cumulative returns across scenarios for the selected portfolios w_1, w_3, w_4, and w_8. The orange line marks the median return, the green dashed line marks the return under the nominal scenario s_2, and the lower tail summarizes outcomes under adverse scenarios.

- Portfolio w_1: Under the nominal scenario s_2, w_1 attains a return lower than w_8 but higher than w_4, and it is close to w_3. In the lower tail, w_1 exhibits the least severe downside outcomes among the four portfolios. Its interquartile range is moderate, and the upper tail includes several positive outliers, indicating strong upside potential in some scenarios.

- Portfolio w_3: Portfolio w_3 achieves a return below w_8 but above w_1 and w_4 in a nominal scenario s_2. In adverse scenarios, the lower tail of w_3 is more negative than that of w_1 and w_8, but less severe than that of w_4. At the same time, w_3 shows a relatively narrow interquartile range, suggesting more stable typical performance across scenarios.
- Portfolio w_4: Portfolio w_4 has the lowest return in a nominal scenario. It also exhibits the most adverse downside outcomes, with a pronounced lower tail and extreme negative outliers, which is consistent with weak return performance in unfavorable scenarios.
- Portfolio w_8: Portfolio w_8 attains the highest return in a nominal scenario. However, its distribution is more dispersed than that of w_3, and its lower tail is more adverse than that of w_1, indicating higher downside exposure despite strong nominal performance.

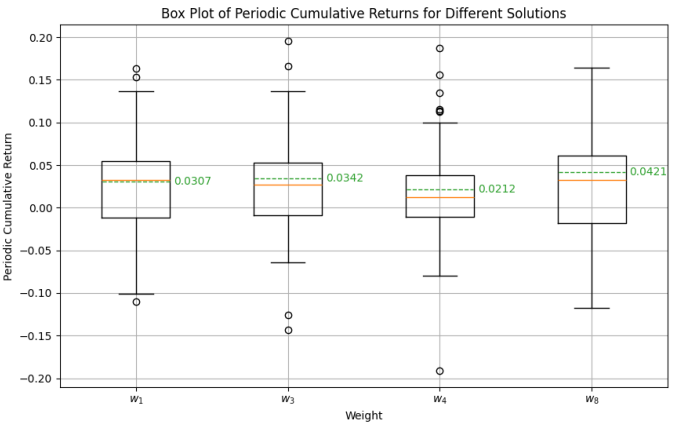
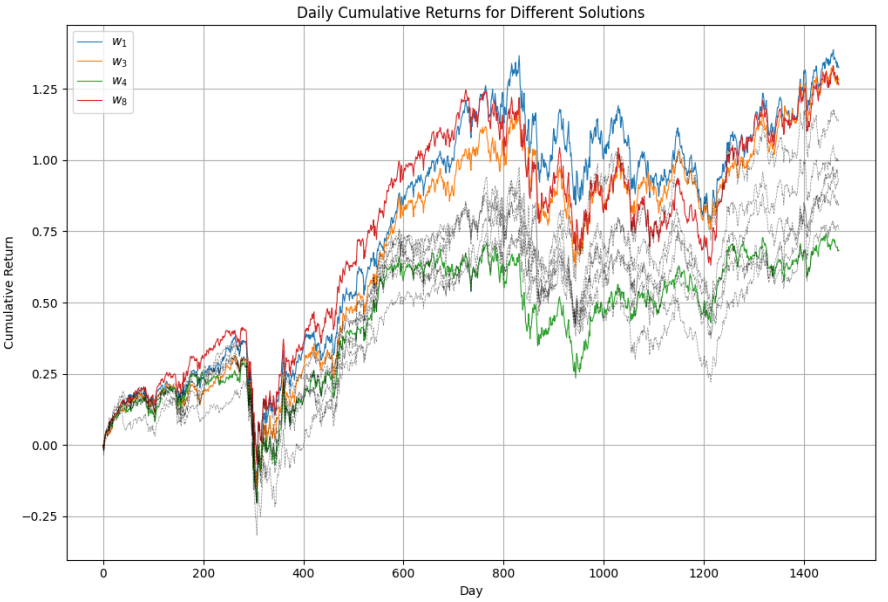


FIGURE 3. Box plot of periodic cumulative returns for selected portfolio w_1, w_3, w_4, and w_8. Outliers are shown as circles, the box represents the interquartile range (IQR), and the whiskers summarize the remaining spread. The orange line indicates the median, and the green dashed line indicates the periodic cumulative returns under the nominal scenario s_2.

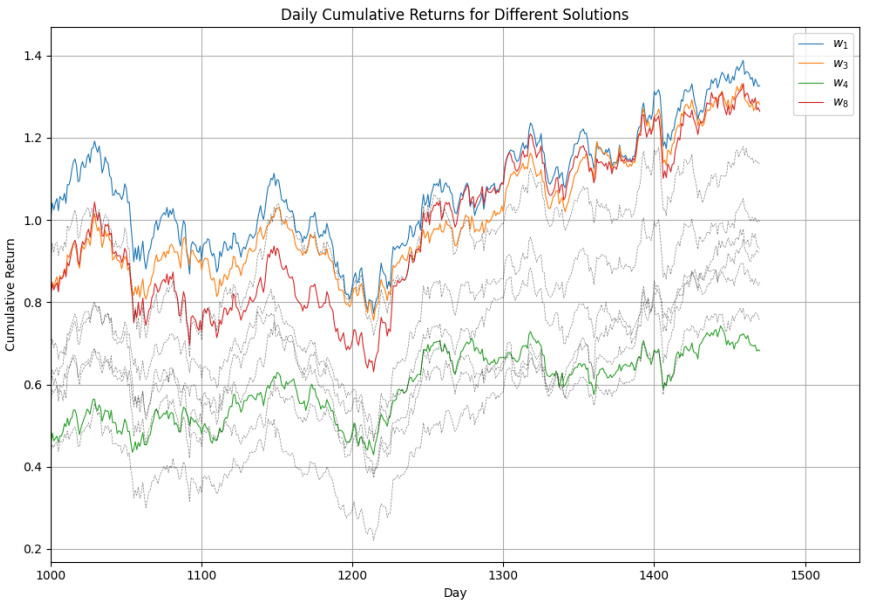
5.6.2. *Performances on return.* Figure 4 provides a visual narrative of the daily cumulative return trajectories for four distinct portfolios: w_1 (blue), w_3 (orange), w_4 (green), and w_8 (red). These are contrasted against the performance of other candidate solutions (dashed gray lines). The plot reveals a clear trade-off between long-term growth and performance stability, highlighting critical events such as the sharp market drawdown around day 300.

The selected portfolios exhibit markedly different risk-return profiles, which underscore the diversity of the candidate solutions:

- Portfolio w_4 (Green): This portfolio follows a distinctly conservative path, successfully mitigating short-term volatility relative to its peers (e.g., during days 400–800). However, its growth remains severely suppressed. Its strategy culminates in a critical failure after day 900, where it suffers a significant and unrecoverable drawdown that relegates it to the lowest terminal wealth among all presented portfolios.



(A) Full evaluation period.



(B) Zoomed view: days 1000–1470.

FIGURE 4. Daily cumulative returns for selected portfolio solutions: (a) full evaluation period; (b) zoomed view for days 1000–1470. The solid colored lines represent the four selected solutions (w_1, w_3, w_4, w_8,), while the gray dashed lines represent the remaining portfolios.

- Portfolios w_1, w_3, and w_8 (Blue, Orange, Red): In contrast, this group constitutes high-performance, high-volatility strategies. All three demonstrate robust growth and successfully navigate the sharp drawdown near day 300 with rapid recoveries. Throughout the investment horizon, they are engaged in a competitive "horse race" for the highest return. Within this race, w_1 and w_8 frequently vie for the lead, while w_3 maintains a strong but slightly lower performance track compared to w_1. Notably, portfolio w_1 achieves the highest peak wealth of any solution around day 850 before experiencing a substantial correction.
- Terminal Performance: In the final phase of the evaluation period (days 1200 onwards), portfolio w_8 emerges with the highest cumulative return, closely followed by w_3 and then w_1. The performance of these three solutions vividly demonstrates the realized success of the more aggressive strategies within this specific historical backtest, contrasting sharply with the nominally risk-averse yet ultimately brittle performance of w_4.

5.7. **Kendall's Tau rank correlation analysis.** To evaluate the consistency of portfolio rankings across different weighting schemes, we employ Kendall's Tau-b rank correlation coefficient [10]. Five distinct weighting schemes, detailed in Table 4, were established to conduct this sensitivity analysis. These schemes range from objective, data-driven weights to various subjective allocations that stress different risk concepts.

TABLE 4. Weighting schemes for sensitivity analysis of criteria.

Set	Weight vector	Rationale / Description
W_0	(0.0787, 0.1350, 0.1017,0.0737,0.3068,0.3041)	Objective weights derived from the Entropy method.
W_1	(0.1667, 0.1667, 0.1667, 0.1667, 0.1667, 0.1667)	Equal weights, representing the Principle of Insufficient Reason.
W_2	(0.1000, 0.1000, 0.2000, 0.2000, 0.2000, 0.2000)	Subjective emphasis on tail and drawdown risk measures (CVaR, CVRG, CDaR, CDaR_Rel).
W_3	(0.2000, 0.2000, 0.2000, 0.2000, 0.1000, 0.1000)	Subjective emphasis on the first four risk measures (MSV, SLPM, CVaR, CVRG).
W_4	(0.2000, 0.2000, 0.1000, 0.1000, 0.2000, 0.2000)	Mixed subjective emphasis, prioritizing MSV, SLPM, CDaR, and CDaR_Rel.
W_5	(0.5000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000)	Strong emphasis on MSV, with minimal weight assigned to the remaining criteria.

Kendall's Tau measures the ordinal association between two rankings by comparing the number of concordant and discordant pairs among all possible item pairs. Given two rankings $R^{(a)}$ and $R^{(b)}$ over N items, Kendall's Tau-b is defined as:

$$\tau = \frac{C - D}{\sqrt{(C + D + T_a)\,(C + D + T_b)}},$$

where:

- C is the number of concordant pairs,
- D is the number of discordant pairs,
- T_a and T_b are the number of ties in rankings a and b, respectively.

A pair (i, j) is concordant if the order of i and j is the same in both rankings, and discordant if the orders are reversed. The coefficient takes values in $[-1, 1]$, where $+1$ indicates identical rankings, 0 indicates no association, and -1 indicates complete inversion.

In this study, Kendall's Tau is computed in two contexts:

(1) Scenario-wise Kendall's Tau correlation matrix: Rankings are derived from all (w, s_q) alternative–scenario combinations, producing a high-resolution correlation assessment across weighting schemes.
(2) Min-over-scenario Kendall's Tau correlation matrix: Each alternative's score is aggregated to its worst-case (minimum) value across scenarios before ranking. This yields a robustness-oriented correlation measure that is more sensitive to weighting scheme differences.

As shown in Table 5, the scenario-wise Kendall's Tau values are consistently high, ranging from 0.9662 to 0.9944, which indicates strong similarity in rankings across schemes under all scenario–alternative combinations. By contrast, the min-over-scenario Kendall's Tau values in Table 6 span from approximately 0.8222 to 1.0000, reflecting greater divergence in rankings when only the worst-case performance is considered.

This dual analysis provides complementary insights. The proposed hybrid robustness–MCDM portfolio framework, which combines entropy-weighted TOPSIS with lightly robust score selection across scenarios, demonstrates reduced sensitivity to perturbations and ensures stability of rankings under varying evaluation conditions.

TABLE 5. Scenario-wise Kendall's Tau correlation matrix.

	W_0	W_1	W_2	W_3	W_4	W_5
W_0	1.0000	0.9778	0.9839	0.9662	0.9831	0.9664
W_1	0.9778	1.0000	0.9931	0.9884	0.9942	0.9875
W_2	0.9839	0.9931	1.0000	0.9818	0.9942	0.9807
W_3	0.9662	0.9884	0.9818	1.0000	0.9830	0.9944
W_4	0.9831	0.9942	0.9942	0.9830	1.0000	0.9833
W_5	0.9664	0.9875	0.9807	0.9944	0.9833	1.0000

TABLE 6. Min-over-scenario Kendall's Tau correlation matrix.

	W_0	W_1	W_2	W_3	W_4	W_5
W_0	1.0000	0.8222	0.8667	0.8222	0.8222	0.8222
W_1	0.8222	1.0000	0.9556	1.0000	1.0000	1.0000
W_2	0.8667	0.9556	1.0000	0.9556	0.9556	0.9556
W_3	0.8222	1.0000	0.9556	1.0000	1.0000	1.0000
W_4	0.8222	1.0000	0.9556	1.0000	1.0000	1.0000
W_5	0.8222	1.0000	0.9556	1.0000	1.0000	1.0000

5.8. **Perturbation analysis of portfolio rankings.** Building on the theoretical framework in Section 3.2, we now present empirical results on the sensitivity of portfolio rankings under criterion-wise and system-level perturbations. The values of $\delta^*_{a\to b,j}$ were obtained numerically by solving the monotone root-finding problem (3.13) using the bracketed toms748 algorithm from SciPy, ensuring both robustness and convergence.

For each scenario s_q and alternative w_i, we compute the minimal perturbation $\delta^*_{a\to b,j}$ required on criterion j to change the ranking position of portfolio w_i relative to its next-higher neighbor. From these values, the differentiated perturbation gap $\Delta^{\text{diff}}_{i,\,a\to b,\,j}$, as defined in (3.13), quantifies the normalized incremental adjustment required for w_i to advance beyond successive competitors.

Aggregating these measures across all criteria and scenarios yields the system differentiated perturbation index, which captures the smallest nonzero adjustment required, across the entire system, for any portfolio to improve its ranking. In our experiment, the system differentiated pertubation was found to be

$$\Delta^{\text{diff}}_{\text{system}} = 2.8630.$$

This highlights the sensitivity of the system to localized changes in performance criteria.

Table 7 reports summary statistics (median, mean, maximum, and standard deviation) of the differentiated perturbation values across the six Sharpe ratio criteria defined in Section 4.4. The results show that differentiated perturbations associated with drawdown-based criteria (CDaR and CDaR_Rel) are substantially larger, whereas volatility- and shortfall-based criteria (MSV, SLPM, CVaR) require comparatively smaller adjustments. This indicates that even a relatively small perturbation can lead to ranking changes under CDaR risk (2.8630). Taken together with the system index, these findings provide a comprehensive perspective on robustness, encompassing both local and global dimensions.

TABLE 7. Summary statistics (median, mean, maximum, and standard deviation) of differentiated perturbation values across the six risk-adjusted performance metrics.

Statistic	MSV	SLPM	CVaR	CVRG	CDaR	CDaR_Rel
Median	0.0967	0.1693	0.3126	0.5252	2.3271	2.2869
Mean	0.0762	0.1319	0.2675	0.4044	1.9402	1.9081
Max	0.1253	0.2047	0.4229	0.6635	2.8630	2.8304
Std. Dev.	0.0365	0.0542	0.1140	0.1988	0.7367	0.7255

6. CONCLUSION

This paper introduced a hybrid robustness–TOPSIS framework for portfolio selection under scenario uncertainty. The framework combines the light-robustness idea with a multi-step MCDM procedure that uses entropy-based weighting, Grey Relational Analysis, and an EDAS-style aggregation. The main contribution is a clear procedure for selecting portfolios that perform well in a nominal market setting while avoiding severe deterioration under adverse scenarios.

An empirical study examined how the proposed framework balances nominal performance and robustness under scenario uncertainty. The results reveal a clear trade-off: portfolios that perform best in the nominal case do not necessarily offer the strongest downside behavior across scenarios. As the relaxation threshold ϵ_m increases, the feasible

set expands beyond the nominally best candidate and the selection shifts toward portfolios with improved worst-case appraisal scores, as reflected in the lower tails of the score distributions.

A comparison across standard solution concepts illustrates how differently nominal performance and downside protection are prioritized under scenario uncertainty. A purely deterministic selection tends to prioritize the nominal outcome and can therefore remain vulnerable when scenarios deteriorate. A stochastic selection typically favors strong average behavior across scenarios, but it does not necessarily guard against extreme downside realizations. In contrast, a minimax robust selection emphasizes worst-case protection and achieves the strongest downside resilience, often at the cost of reduced nominal performance. The proposed lightly robust framework sits between these extremes: by allowing a controlled relaxation of nominal optimality, it systematically shifts the choice toward portfolios with improved worst-case behavior while avoiding excessive conservatism. This mechanism yields a transparent and practical way to balance nominal efficiency and downside protection under scenario uncertainty.

The risk-adjusted evidence is consistent with the proposed framework. Across the considered downside-risk measures, the scenario-wise Sharpe ratio distributions largely follow the appraisal-based ranking, especially with respect to nominal performance and lower-tail behavior in adverse scenarios. While some deviations occur for particular risk definitions, the overall pattern supports using multiple risk measures in the MCDM step, since aggregating Sharpe ratios reduces reliance on any single risk specification and improves ranking stability under scenario uncertainty.

The return-based results further illustrate that alternative solution concepts induce different risk–return profiles. Approaches that emphasize nominal performance can deliver higher average returns but may remain vulnerable in unfavorable scenarios, whereas worst-case robust approaches offer stronger protection at the cost of conservatism. By controlling the nominal–robustness trade-off, the proposed framework provides an intermediate choice that is not necessarily best by returns alone, but is more reliable when return and downside risk are assessed jointly under scenario uncertainty.

The Kendall's Tau rank correlation results further indicate that the proposed ranking is stable across weighting specifications. The consistently high correlations across the five weighting schemes suggest that the portfolio ordering is not overly sensitive to alternative views on the relative importance of the risk criteria.

Several directions for future research remain. First, the current study evaluates static portfolio allocations; extending the framework to dynamic rebalancing settings would allow the selection to adapt to evolving market conditions. Second, the modular structure of the method enables the investigation of alternative risk measures and MCDM aggregation operators to test robustness across different modeling choices. Finally, applying this methodology to different asset classes, such as cryptocurrencies or international bonds, would be a valuable extension to validate its applicability beyond the equity markets.

Acknowledgements

This work was supported by Naresuan University (NU), and National Science, Research and Innovation Fund (NSRF), Grant NO. R2567B076.

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