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New Mann type iterative methods with a general resolvent on geodesic spaces

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ABSTRACT. Approximation methods to a minimizer of a convex function have been studied by a lot of mathematicians for a long time. In this paper, we investigate a convergence theorem to a minimizer of a given convex function by using the Mann type iterative method and a resolvent for convex functions on geodesic spaces. Moreover, we apply the resolvents proposed by Kajimura, Kimura and Kohsaka quite recently to our main result in this paper.

1. INTRODUCTION

As is well known, approximation methods for a fixed point of a given nonlinear mapping and a minimizer of a given convex function have been studied in various ways by many researchers, respectively. The Mann type iterative method is one of the most celebrated methods to find a fixed point of nonlinear mappings. In 1953, it was proposed by Mann [23] to approximate a fixed point of nonexpansive mappings in a Hilbert space. In 2008, Dhompongsa and Panyanak [8] extended it to a complete CAT(0) space. They generated the Mann iteration sequence by

$$x_{n+1} = \alpha_n T x_n \oplus (1-\alpha_n)x_n$$

for all $n \in \mathbb{N}$, where T is a nonexpansive mapping on a CAT(0) space and $\{\alpha_n\} \subset [0,1]$. For more results about the Mann type iterative method, see, for instance, [1, 15, 26]. In 1967, Halpern [10] proposed the strong convergence method to find a fixed point of nonlinear mappings, see also [31].

Resolvents for convex functions play an important role to find a minimizer of a given convex function. For $\kappa \in \{1, 0\}$, let X_κ be an admissible complete CAT(κ) space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. In 1998, Mayer [25] proposed the resolvent of λf by

$$J_{\lambda f} x = \operatorname*{argmin}_{y \in X_0} \left\{ f(y) + \frac{1}{2\lambda} d(y,x)^2 \right\}$$

in a complete CAT(0) space, where λ is a positive real number. By using this resolvent, Bačák [2] showed that the sequence

$$x_{n+1} = J_{\lambda_n} x_n$$

is Δ-convergent to an element of $\operatorname{argmin}_X f$, where $x_1 \in X_0$ and $\{\lambda_n\}$ a sequence of positive real numbers satisfying that $\sum_{n=1}^{\infty} \lambda_n = \infty$. This method is called the proximal point algorithm, and it was proposed by Rockafellar [28] for finding a zero-point of a

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monotone operator in a Hilbert space in 1976, see also [4, 24]. For more results of zero-points problems, see, for example, [6, 7, 11, 16, 27, 30]. In 2016, Kimura and Kohsaka proved the following theorem:

Theorem 1.1 ([19]). *Let X be a complete* CAT(0) *space, f a proper lower semicontinuous convex function from X into $]-\infty,\infty]$, $\{\lambda_n\}$ a sequence of positive real numbers and $\{\alpha_n\}$ a sequence of $[0,1[$ satisfying that $\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n=\infty$. Generate a sequence $\{x_n\}$ of X by*

$$x_1 \in X \text{ and } x_{n+1}=\alpha_n x_n \oplus (1-\alpha_n)J_{\lambda_n f}x_n$$

for all $n\in\mathbb{N}$. Then the following hold:

(i) $\operatorname{argmin}_X f$ *is nonempty if and only if $\{J_{\lambda_n f}x_n\}$ is bounded;*

(ii) if $\operatorname{argmin}_X f$ *is nonempty and* $\sup_n \alpha_n<1$*, then both $\{x_n\}$ and $\{J_{\lambda_n f}x_n\}$ are Δ-convergent to an element of* $\operatorname{argmin}_X f$.

This theorem is a generalization of the proximal point algorithm in a complete CAT(0) space. On the other hand, some resolvents of λf were proposed in an admissible complete CAT(1) space for $\lambda>0$. For instance, Kimura and Kohsaka [18] defined a resolvent of λf by

$$Q_{\lambda f}x=\operatorname*{argmin}_{y\in X_1}\left\{f(y)+\frac{1}{\lambda}\tan d(y,x)\sin d(y,x)\right\}.$$

They also showed the following theorem:

Theorem 1.2 ([21]). *Let X be an admissible complete* CAT(1) *space, f a proper lower semicontinuous convex function from X into $]-\infty,\infty]$, $\{\lambda_n\}$ a sequence of positive real numbers and $\{\alpha_n\}$ a sequence of $[0,1[$ satisfying that $\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n=\infty$. Generate a sequence $\{x_n\}$ of X by*

$$x_1 \in X \text{ and } x_{n+1}=\alpha_n x_n \oplus (1-\alpha_n)Q_{\lambda_n f}x_n$$

for all $n\in\mathbb{N}$. Then the following hold:

(i) $\operatorname{argmin}_X f$ *is nonempty if and only if*

$$\inf_{y\in X}\limsup_{n\to\infty} d(Q_{\lambda_n f}x_n,y)<\pi/2 \text{ and } \sup_n d(Q_{\lambda_n}x_n,x_n)<\pi/2;$$

(ii) if $\operatorname{argmin}_X f$ *is nonempty and* $\sup_n \alpha_n<1$*, then both $\{x_n\}$ and $\{Q_{\lambda_n f}x_n\}$ are Δ-convergent to an element of* $\operatorname{argmin}_X f$.

We notice that Theorems 1.1 and 1.2 need an additional condition. In their proofs, they use the following technique:

"Since $-\infty<\inf f(X)\leqq f(x_{n+1})\leqq f(x_n)$,
$\{f(x_n)\}$ is convergent to some real number."

However, we do not know whether $f(x_n)$ belongs to $\mathbb{R}$ or not. Therefore, we need an assumption which generates $f(x_n)\in\mathbb{R}$ for some $n\in\mathbb{N}$.

The aim of this paper is to generalize Theorems 1.1 and 1.2 in CAT(κ) spaces, and we modify their assumptions. Furthermore, we apply the resolvents

$$S_{\lambda f}x=\operatorname*{argmin}_{y\in X_1}\left\{f(y)+\frac{1}{\lambda}(1-\cos d(y,x)-\log(\cos d(y,x)))\right\}$$

and

$$T_{\lambda f}x=\operatorname*{argmin}_{y\in X_{-1}}\left\{f(y)+\frac{1}{\lambda}(\cosh d(y,x)-1+\log(\cosh d(y,x)))\right\},$$

which we proposed quite recently in the paper [14] to our main result.

2. PRELIMINARIES

Throught this paper, we denote the set of all real numbers by $\mathbb{R}$, the set of all positive integers by $\mathbb{N}$ and the set of all minimizers of a convex function f from a set X into $]-\infty,\infty]$ by $\operatorname{argmin}_X f$, respectively. Let X be a metric space. We define an asymptotic center of a sequence $\{x_n\}$ of X by

$$\left\{x \in X \;\middle|\; \inf_{y\in X} \limsup_{n\to\infty} d(x_n,y) = \limsup_{n\to\infty} d(x_n,x)\right\}$$

which is denoted by $\mathcal{A}(\{x_n\})$. A sequence $\{x_n\}$ is Δ-convergent to $x_\infty \in X$ if the asymptotic center of every subsequence of $\{x_n\}$ consists of x_∞. A set $\omega_\Delta(\{x_n\})$ is defined by

$$\{x \in X \mid \text{there exists } \{x_{n_i}\} \subset \{x_n\} \text{ such that } \{x_{n_i}\} \text{ is } \Delta\text{-convergent to } x\}.$$

A function f from X into $]-\infty,\infty]$ is said to be

- proper if there exists $x \in X$ such that $f(x) < \infty$;
- lower semicontinuous if
$$f(x_\infty) \leqq \liminf_{n\to\infty} f(x_n)$$
for every sequence $\{x_n\}$ of X which is convergent to x_∞;
- Δ-lower semicontinuous if
$$f(x_\infty) \leqq \liminf_{n\to\infty} f(x_n)$$
for every sequence $\{x_n\}$ of X which is Δ-convergent to x_∞.

For $x, y \in X$, a mapping γ_{xy} from $[0, d(x,y)]$ into X is called a geodesic connecting $x, y \in X$ if the following properties hold:

- $\gamma_{xy}(0) = x$ and $\gamma_{xy}(d(x,y)) = y$;
- $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s-t|$ for all $s, t \in [0, d(x,y)]$.

For $D \in]0,\infty]$, a metric space X is called a D-uniquely geodesic space if any two points $x, y \in X$ which satisfy $d(x,y) < D$ have a unique geodesic γ_{xy}. Let X be a D-uniquely geodesic space. For each $x, y \in X$ with $d(x,y) < D$, we define a convex combination between x and y by

$$tx \oplus (1-t)y = \gamma_{xy}((1-t)d(x,y))$$

for all $t \in [0,1]$. A function f from X into $]-\infty,\infty]$ is said to be convex if

$$f(tx \oplus (1-t)y) \leqq tf(x) + (1-t)f(y)$$

for all $x, y \in X$ and $t \in]0,1[$.

For $\kappa \in \mathbb{R}$, a two dimensional model space with curvature κ is defined by

$$M_\kappa^2 = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\kappa > 0); \\ \mathbb{E}^2 & (\kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\kappa < 0), \end{cases}$$

where $\mathbb{S}^2$ is the two-dimensional unite sphere on $\mathbb{R}^3$, $\mathbb{E}^2$ is two-dimensional Euclidean space and $\mathbb{H}^2$ is the two-dimensional hyperbolic space. We denote the diameter of M_κ^2 by D_κ, that is,

$$D_\kappa = \begin{cases} \pi/\sqrt{\kappa} & (\kappa > 0); \\ \infty & (\kappa \leqq 0). \end{cases}$$

We know that M^2_κ is a D_κ-uniquely geodesic space. Let X be a D_κ-uniquely geodesic space. For each $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, a geodesic triangle $\triangle(x,y,z)$ of X is defined by

$$\triangle(x,y,z) = \gamma_{xy}([0, d(x,y)]) \cup \gamma_{yz}([0, d(y,z)]) \cup \gamma_{zx}([0, d(z,x)]).$$

A geodesic triangle $\overline{\triangle}(x', y', z')$ on M^2_κ is called a comparison triangle of $\triangle(x,y,z) \subset X$ if the length of corresponding edges are identical. For $p \in \triangle(x,y,z)$, a point $\bar{p} \in \overline{\triangle}(x', y', z')$ is called a comparison point of p if at least one of the following conditions holds:

- if $p = tx \oplus (1-t)y$, then $\bar{p} = t\bar{x} \oplus (1-t)\bar{y}$;
- if $p = ty \oplus (1-t)z$, then $\bar{p} = t\bar{y} \oplus (1-t)\bar{z}$;
- if $p = tz \oplus (1-t)x$, then $\bar{p} = t\bar{z} \oplus (1-t)\bar{x}$,

for $t \in [0,1]$. X is called a CAT(κ) space if $d(p,q) \leqq d(\bar{p}, \bar{q})$ for all $\triangle(x,y,z)$ and $p, q \in \triangle(x,y,z)$, for more details, see [3, 5]. A sequence $\{x_n\}$ of a CAT(κ) space X is said to be geodesically bounded if

$$\inf_{y \in X} \limsup_{n \to \infty} d(x_n, y) < \frac{D_\kappa}{2}$$

holds. If $\kappa \leqq 0$, then the geodesical boundedness coincides with the usual boundedness in metric spaces since $D_\kappa = \infty$. A CAT(κ) space is said to be admissible if $d(x,y) < D_\kappa/2$ for all $x, y \in X$. In an admissible complete CAT(κ) space X, it is well known that every proper lower semicontinuous convex function is Δ-lower semicontinuous, see [3, 18]. Moreover, the following properties are well known:

Lemma 2.1 ([9, 22]). *Let X be an admissible complete* CAT(κ) *space and $\{x_n\}$ a geodesically bounded sequence of X. Then $\mathcal{A}(\{x_n\})$ consists of one point and $\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.2 ([17]). *Let X be an admissible complete* CAT(κ) *space and $\{x_n\}$ a geodesically bounded sequence of X. If $\{d(z, x_n)\}$ is convergent for each $z \in \omega_\Delta(\{x_n\})$, then $\{x_n\}$ is Δ-convergent to an element of X.*

For $\kappa \in \mathbb{R}$, a function c_κ from $[0, D_\kappa/2[$ into $[0, \infty[$ is defined by

$$c_\kappa(t) = \begin{cases} \dfrac{1}{\kappa}(1 - \cos(\sqrt{\kappa}t)) & (\kappa > 0); \\ \dfrac{1}{2}t^2 & (\kappa = 0); \\ \dfrac{1}{-\kappa}(\cosh(\sqrt{-\kappa}t) - 1) & (\kappa < 0) \end{cases}$$

for all $t \in [0, D_\kappa/2[$. Further, we define $c_\kappa(D_\kappa/2)$ by $\lim_{t \to D_\kappa/2} c_\kappa(t)$. We notice that c_κ is continuous, strictly convex and strictly increasing on $[0, D_\kappa/2[$, and $c_\kappa(0) = 0$. Moreover, it follows that

$$c'_\kappa(t) = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t) & (\kappa > 0); \\ t & (\kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t) & (\kappa < 0) \end{cases}$$

and

$$c''_\kappa(t) = \begin{cases} \cos(\sqrt{\kappa}t) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh(\sqrt{-\kappa}t) & (\kappa < 0). \end{cases}$$

The following lemmas are known.

Lemma 2.3 ([12, 19, 20]). *For $\kappa \in \mathbb{R}$, let X be an admissible complete $\mathrm{CAT}(\kappa)$ space, $\{z_k\}$ a geodesically bounded sequence of X, $\{\beta_k\}$ a sequence of positive real numbers with $\sum_{k=1}^{\infty} \beta_k = \infty$ and*

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{k=1}^{n} \beta_k} \sum_{k=1}^{n} \beta_k c_\kappa(d(y, z_k))$$

for each $y \in X$. Then $\operatorname{argmin}_X g$ consists of one point.

Lemma 2.4 ([13]). *For $\kappa \in \mathbb{R}$, let X be an admissible $\mathrm{CAT}(\kappa)$ space, f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ and φ a strictly increasing function from $[0, c_\kappa(D_\kappa/2)[$ into $[0, \infty[$ of class C^1 and λ a positive real number. Assume that*

$$R_{\lambda f} x = \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{\lambda} \varphi(c_\kappa d(y, x)) \right\}$$

is a singleton for each $x \in X$. Put $C_{\eta,z} = c_\kappa d(R_{\eta f} z, z)$ for $\eta > 0$ and $z \in X$. Then

$$\begin{aligned} &\lambda(f(R_{\lambda f}x) - f(y)) \\ &\leqq \frac{\varphi'(C_{\lambda,x}) d(R_{\lambda f}x, y)}{c'_\kappa d(R_{\lambda f}x, y)} (c_\kappa d(x, y) - C_{\lambda,x} c''_\kappa d(R_{\lambda f}x, y) - c_\kappa d(R_{\lambda f}x, y)) \end{aligned}$$

for all $x, y \in X$ with $R_{\lambda f}x \neq y$ and

$$\begin{aligned} &(\lambda\varphi'(C_{1,y}) C_{\mu,y} + \varphi'(C_{\lambda,x}) C_{\lambda,x})) c''_\kappa(D) \\ &\quad + (\lambda\varphi'(C_{1,y}) + \varphi'(C_{\lambda,x})) c_\kappa(D) \\ &\qquad \leqq \lambda\varphi'(C_{1,y}) c_\kappa d(R_{\lambda f}x, y) + \varphi'(C_{\lambda,x}) c_\kappa d(x, R_f y) \end{aligned}$$

for all $x, y \in X$, where $D = d(R_{\lambda f}x, R_f y)$. Moreover,

$$\operatorname*{argmin}_X f = \mathcal{F}(R_{\lambda_n f})$$

holds for all $n \in \mathbb{N}$.

3. Mann type algorithm

In this section, we prove a main theorem which is a generalization of Theorems 1.1 and 1.2. Further, we obtain a convergence theorem to a convex function by applying the resolvents $S_{\lambda f}$ and $T_{\lambda f}$ to our main result.

Theorem 3.3. *Let κ, X, f and φ be the same as in Lemma 2.4. Let $\{\lambda_n\}$ be a sequence of positive real numbers and $\{\alpha_n\}$ a sequence of $[0, 1[$ satisfying that $\alpha_1 = 0$ and $\sum_{n=1}^{\infty}(1 - \alpha_n)\lambda_n = \infty$. Assume that*

$$R_{\lambda_n f} x = \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{\lambda_n} \varphi(c_\kappa d(y, x)) \right\}$$

is a singleton for each $x \in X$. Generate a sequence $\{x_n\}$ of X by

$$\begin{aligned} &x_1 \in X; \\ &z_n = R_{\lambda_n f} x_n; \\ &x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) z_n \end{aligned}$$

for $n \in \mathbb{N}$. Then the following hold:

(i) If $\operatorname{argmin}_X f$ is nonempty then both

$$d(z_n, x_n) \leqq d(x_n, p) \text{ and } d(z_n, p) \leqq d(x_n, p)$$

hold for all $p \in \operatorname{argmin}_X f$;

(ii) $\operatorname{argmin}_X f$ *is nonempty if and only if* $\sup_n d(z_n, x_n) < D_\kappa/2$ *and* $\{z_n\}$ *is geodesically bounded;*

(iii) if $\operatorname{argmin}_X f$ *is nonempty and* $\sup_n \alpha_n < 1$*, then both* $\{x_n\}$ *and* $\{z_n\}$ *are* Δ*-convergent to an element of* $\operatorname{argmin}_X f$*.*

Proof. Put $C_{\eta,z} = c_\kappa d(R_{\eta f} z, z)$ for $\eta > 0$ and $z \in X$. We first show (i). Let $p \in \operatorname{argmin}_X f$. By Lemma 2.4, we have

$$\begin{aligned} 0 &\leqq \lambda_n(f(z_n) - f(p)) \\ &\leqq c_\kappa d(x_n, p) - C_{\lambda_n, x_n} c''_\kappa d(z_n, p) - c_\kappa d(z_n, p) \end{aligned}$$

and thus the following inequalities hold:

$$\begin{cases} d(x_{n+1}, x_n)^2 + d(z_n, p)^2 \leqq d(x_n, p)^2 & (\kappa = 0); \\ \cos(\sqrt{\kappa} d(x_{n+1}, x_n)) \cos(\sqrt{\kappa} d(z_n, p)) \geqq \cos(\sqrt{\kappa} d(x_n, p)) & (\kappa > 0); \\ \cosh(\sqrt{-\kappa} d(x_{n+1}, x_n)) \cosh(\sqrt{-\kappa} d(z_n, p)) \leqq \cos(\sqrt{-\kappa} d(x_n, p)) & (\kappa < 0). \end{cases}$$

From the admissibility of X, both

$$d(x_n, z_n) \leqq d(x_n, p) \text{ and } d(z_n, p) \leqq d(x_n, p)$$

hold.

We next show the the only if part of (ii). Let $p \in \operatorname{argmin}_X f$. By (i), we obtain

$$\begin{aligned} c_\kappa d(x_{n+1}, p) &\leqq \alpha_n c_\kappa d(x_n, p) + (1 - \alpha_n) c_\kappa d(z_n, p) \\ &\leqq \alpha_n c_\kappa d(x_n, p) + (1 - \alpha_n) c_\kappa d(x_n, p) = c_\kappa d(x_n, p) \end{aligned} \tag{3.1}$$

and hence

$$d(x_{n+1}, p) \leqq d(x_n, p) \leqq \cdots \leqq d(x_1, p) < D_\kappa/2. \tag{3.2}$$

Therefore, it follows from (i) that

$$d(z_n, p) \leqq d(x_n, p) \leqq d(x_1, p) < D_\kappa/2 \tag{3.3}$$

and that $\{z_n\}$ is geodesically bounded. Moreover, we have

$$d(z_n, x_n) \leqq d(x_n, p) \leqq d(x_1, p) < D_\kappa/2 \tag{3.4}$$

and hence $\sup_n d(z_n, x_n) < D_\kappa/2$. We show the if part of (i). Put

$$\beta_n = \frac{(1 - \alpha_n)\lambda_n}{\varphi'(C_{\lambda_n, x_n})} \text{ and } \sigma_n = \sum_{k=1}^{n} \beta_k$$

for $n \in \mathbb{N}$. By the assumption, there exists a real number $d \in]0, D_\kappa/2[$ such that $d = \sup_m d(z_m, x_m)$. Furthermore, since $\varphi' \circ c_\kappa$ is a continuous function on $[0, d]$, there exists a real number M such that $\max_{t \in [0,d]} \varphi' \circ c_\kappa(t) = M > 0$. Therefore, we obtain

$$\sigma_n = \sum_{k=1}^{n} \beta_k \geqq \sum_{k=1}^{n} \frac{(1 - \alpha_n)\lambda_n}{M} \to \infty$$

as $n \to \infty$. From Lemma 2.3, the function

$$g(y) = \limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k c_\kappa d(y, z_k)$$

has a unique minimizer $p \in X$. By Lemma 2.4, we get

$$\begin{aligned}&\frac{\lambda_k}{\varphi'(C_{\lambda_k,x_k})}c_\kappa d(z_k, R_f p)\\ &\leqq \frac{\lambda_k}{\varphi'(C_{\lambda_k,x_k})}c_\kappa d(z_k, p) + \frac{1}{\varphi'(C_{1,p})}(c_\kappa d(x_k, R_f p) - c_\kappa d(z_k, R_f p))\end{aligned}$$

and hence

$$\begin{aligned}&\beta_k c_\kappa d(z_k, R_f p)\\ &\leqq \beta_k c_\kappa d(z_k, p) + \frac{1-\alpha_k}{\varphi'(C_{1,p})}(c_\kappa d(x_k, R_f p) - c_\kappa d(z_k, R_f p))\\ &\leqq \beta_k c_\kappa d(z_k, p) + \frac{1}{\varphi'(C_{1,p})}((1-\alpha_k)c_\kappa d(x_k, R_f p) - (1-\alpha_k)c_\kappa d(z_k, R_f p)).\end{aligned}$$

On the other hand, we know that

$$c_\kappa d(x_{k+1}, R_f p) \leqq \alpha_k c_\kappa d(x_k, R_f p) + (1-\alpha_k)c_\kappa d(z_k, R_f p).$$

Consequently, we get

$$\begin{aligned}&\beta_k c_\kappa d(z_k, R_f p)\\ &\leqq \beta_k c_\kappa d(z_k, p) + \frac{1}{\varphi'(C_{1,p})}(c_\kappa d(x_k, R_f p) - c_\kappa d(x_{k+1}, R_f p)),\end{aligned}$$

and thus

$$\begin{aligned}&\frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k c_\kappa d(z_k, R_f p)\\ &\leqq \frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k c_\kappa d(z_k, p) + \frac{1}{\sigma_n\varphi'(C_{1,p})}(c_\kappa d(x_1, R_f p) - c_\kappa d(x_{n+1}, R_f p))\\ &\leqq \frac{1}{\sigma_n}\sum_{k=1}^{n}\beta_k c_\kappa d(z_k, p) + \frac{c_\kappa d(x_1, R_f p)}{\sigma_n\varphi'(C_{1,p})}.\end{aligned}$$

Taking the upper limit as n tends to ∞, we get $g(R_f p) \leqq g(p)$. Since p is a unique minimizer of g, we have $R_f p = p \in \mathcal{F}(R_f) = \operatorname{argmin}_X f$.

We finally show (iii). Since $\alpha_1 = 0$, we have $x_2 = R_{\lambda_1} x_1$ and thus $f(x_2) \in \mathbb{R}$. Let $n \in \mathbb{N}$ satisfying that $f(x_n) \in \mathbb{R}$. Since $\operatorname{argmin}_X f$ is nonempty, it follows from the definition of $R_{\lambda_n f}$ that

$$-\infty < \inf f(X) \leqq f(z_n) \leqq f(z_n) + \frac{1}{\lambda_n}\varphi(c_\kappa d(z_n, x_n)) - \frac{1}{\lambda_n}\varphi(0) \leqq f(x_n) \tag{3.5}$$

and hence

$$-\infty < \inf f(X) \leqq f(x_{n+1}) \leqq \alpha_n f(x_n) + (1-\alpha_n)f(z_n) \leqq f(x_n). \tag{3.6}$$

By induction, $f(x_n) \in \mathbb{R}$ for all $n \geqq 2$. Therefore, the sequence $\{f(x_n)\}$ is convergent to a real number γ, and $\{f(z_n)\}$ is a bounded sequence. Let $\{f(z_{n_i})\} \subset \{f(z_n)\}$. Since $\sup_n \alpha_n < 1$, we can take a subsequence $\{\alpha_{n_{i_j}}\}$ of $\{\alpha_{n_i}\}$ which is convergent to $\delta \in [0, 1[$. By the inequalities (3.5) and (3.6), we have

$$\frac{1}{1-\alpha_{n_{i_j}}}(f(x_{n_{i_j}+1}) - \alpha_{n_{i_j}} f(x_{n_{i_j}})) \leqq f(z_{n_{i_j}}) \leqq f(x_{n_{i_j}})$$

and thus both $\{f(x_n)\}$ and $\{f(z_n)\}$ are convergent to γ. Let $p \in \operatorname{argmin}_X f$. By (3.2), the sequence $\{d(x_n, p)\}$ is convergent to $\beta \in [0, D_\kappa/2[$. By Lemma 2.4, there exists a positive real number L such that

$$\begin{aligned}(1-\alpha_n)\lambda_n(f(z_n)-f(p)) &\leqq L((1-\alpha_n)c_\kappa d(x_n,p)-(1-\alpha_n)c_\kappa d(z_n,p))\\ &\leqq L(c_\kappa d(x_n,p)-c_\kappa d(x_{n+1},p)).\end{aligned}$$

Consequently, we obtain

$$\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n(f(z_n)-f(p)) \leqq L(c_\kappa d(x_1,p)-c_\kappa(\beta)) \in \mathbb{R}.$$

Since $\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n = \infty$, we get

$$\liminf_{n\to\infty}(f(z_n)-f(p)) = 0.$$

Since both $\{f(x_n)\}$ and $\{f(z_n)\}$ are convergent to γ, it follows

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(z_n) = f(p).$$

Let $z \in \omega_\Delta(\{x_n\})$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is Δ-convergent to z. Since f is Δ-lower semicontinuous, we get

$$f(z) \leqq \liminf_{i\to\infty} f(x_{n_i}) = f(p).$$

This implies that $z \in \operatorname{argmin}_X f$. Since $\{d(x_n, z)\}$ is convergent to some real number, it follows from Lemma 2.2 that $\{x_n\}$ is Δ-convergent to $x_\infty \in X$. By using the same technique, we obtain $x_\infty \in \operatorname{argmin}_X f$. On the other hand, by (3.4), $\{d(z_n, x_n)\}$ is bounded. Let $\{d(z_{n_i}, x_{n_i})\} \subset \{d(z_n, x_n)\}$ arbitrarily. From (3.3) and $\sup_n \alpha_n < 1$, we can take subsequences $\{d(z_{n_{i_j}}, p)\} \subset \{d(z_{n_i}, p)\}$ and $\{\alpha_{n_{i_j}}\} \subset \{\alpha_{n_i}\}$ which satisfy

$$\lim_{j\to\infty} d(z_{n_{i_j}}, p) = d \in [0, D_\kappa/2[\text{ and } \lim_{j\to\infty} \alpha_{n_{i_j}} = \alpha \in [0, 1[\,.$$

It follows from Lemma 2.4 and (3.1) that

$$\begin{aligned}0 \leqq c_\kappa d(z_{n_{i_j}}, x_{n_{i_j}}) &\leqq \frac{c_\kappa d(x_{n_{i_j}}, p) - c_\kappa d(z_{n_{i_j}}, p)}{c''_\kappa d(z_{n_{i_j}}, p)}\\ &\leqq \frac{c_\kappa d(x_{n_{i_j}}, p) - \dfrac{c_\kappa d(x_{n_{i_j}+1}, p) - \alpha_{n_{i_j}} c_\kappa d(x_{n_{i_j}}, p)}{1-\alpha_{n_{i_j}}}}{c''_\kappa d(z_{n_{i_j}}, p)}\\ &\to \frac{c_\kappa(\beta) - \dfrac{c_\kappa(\beta) - \alpha c_\kappa(\beta)}{1-\alpha}}{c''_\kappa(d)} = 0\end{aligned}$$

as j tends to ∞. This immediately implies that $\lim_{n\to\infty} d(z_n, x_n) = 0$. Since $\{x_n\}$ is Δ-convergent to x_∞, $\{z_n\}$ is also Δ-convergent to x_∞. □

Let X be an admissible complete CAT(1) space. We define a function φ from $[0, 1[$ into $[0, \infty[$ by $\varphi_1(a) = a - \log(1-a)$ for all $a \in [0, 1[$. Then we can express the resolvent $S_{\lambda f}$ by

$$S_{\lambda f}x = \operatorname*{argmin}_{y\in X}\left\{f(y) + \frac{1}{\lambda}\varphi_1(c_1 d(y, x))\right\}$$

for $x \in X$. Note that φ_1 is a strictly increasing function on $[0, 1[$ of class C^1. On the other hand, in a complete CAT(-1) space X, define a function φ_{-1} from $[0, \infty[$ into $[0, \infty[$ by

$\varphi_{-1}(a) = a + \log(1+a)$ for all $a \in [0,\infty[$. It is also strictly increasing function on $[0,\infty[$ of class C^1. Moreover, we can express $T_{\lambda f}$ by

$$T_{\lambda f}x = \operatorname*{argmin}_{y\in X}\left\{f(y) + \frac{1}{\lambda}\varphi_{-1}(c_{-1}d(y,x))\right\}$$

for $x \in X$. In the paper [14], we show the well-defindeness of $S_{\lambda f}$ and $T_{\lambda f}$ as single-valued mappings. Thus, it follows from Theorem 3.3 that the following theorem:

Theorem 3.4. *For $\kappa \in \{1,-1\}$, let X_κ be an admissible complete $\mathrm{CAT}(\kappa)$ space, f a proper lower semicontinuous convex function, $\{\lambda_n\}$ a sequence of positive real numbers and $\{\alpha_n\}$ a sequence of $[0,1[$ satisfying $\alpha_1 = 0$ and $\sum_{n=1}^{\infty}(1-\alpha_n)\lambda_n = \infty$. Generate a sequence $\{x_n\}$ of X_κ by $x_1 \in X_\kappa$ and*

$$\begin{cases} x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)S_{\lambda_n f}x_n & (\kappa = 1);\\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T_{\lambda_n f}x_n & (\kappa = -1) \end{cases}$$

for all $n \in \mathbb{N}$. If $\operatorname{argmin}_X f$ is nonempty and $\sup_n \alpha_n < 1$, then $\{x_n\}$, $\{S_{\lambda_n f}x_n\}$ and $\{T_{\lambda_n f}x_n\}$ are Δ-convergent to an element of $\operatorname{argmin}_X f$.

4. Application in a Hilbert space

In this section, we give an application in a Hilbert space. By using Theorem 3.3, we get the following theorem.

Theorem 4.5. *Let H be a Hilbert space, f a proper lower semicontinuous convex function from H into $]-\infty,\infty]$ with $\operatorname{argmin}_H f \neq \emptyset$, $p \geqq 2$ and*

$$R_n x = \operatorname*{argmin}_{y\in H}\left\{f(y) + \frac{1}{n}\|y-x\|^p\right\}$$

for $n \in \mathbb{N}$. Generate a sequence $\{x_n\}$ of H by $x_1 \in X$ and

$$x_{n+1} = \left(1 - \frac{1}{n}\right)x_n + \frac{1}{n}R_n x_n.$$

Then $\{x_n\}$ is weakly convergent to an element of $\operatorname{argmin}_H f$.

Proof. Let $\varphi(t) = 2^{p/2}t^{p/2}$ for all $t \in [0,\infty[$. Then we know that φ is strictly increasing and φ' is continuous on $[0,\infty[$. Moreover, we notice that

$$R_n x = \operatorname*{argmin}_{y\in H}\left\{f(y) + \frac{1}{n}\varphi\left(\frac{1}{2}\|y-x\|^2\right)\right\}$$

for $x \in X$. We also know that the sequence $\{x_n\}$ is well defined. On the other hand,

$$\sum_{n=1}^{\infty}\frac{1}{n}\cdot n = \infty.$$

Therefore, by Theorem 3.3, we get the conclusion. □

We remark that the resolvent used in Theorem 4.5 is deduced to the classical one by putting $p = 2$.

5. Conclusion

In this paper, we studied Mann type iterative methods with a general resolvent for convex functions on admissible complete CAT(κ) spaces. We established convergence results for approximating a minimizer of a given convex function by using the Mann iterative scheme with a resolvent operator. Our results extend several known convergence theorems for proximal type algorithms on geodesic spaces.

In this work, we considered a general class of resolvents defined by perturbation functions. The class of perturbation functions used in this paper is sufficiently broad and includes many resolvents that have been studied in earlier works as special cases. From this viewpoint, our framework provides a unified way to treat both classical type resolvents and more recently proposed resolvents on geodesic spaces with curvature bounded above.

Another important aspect of our analysis is that it is carried out in admissible complete CAT(κ) spaces with an arbitrary real curvature bound κ. This makes it possible to handle spaces of nonpositive curvature, flat spaces, and spaces of positive curvature in a single setting. As a consequence, many existing results on Mann type iterations and convex minimization problems can be covered within our framework.

Finally, we applied the main theorem to several concrete examples. These include resolvents on CAT(1) and CAT(-1) spaces, as well as an application in a Hilbert space. Through these examples, we showed that the proposed method can be used in different geometric settings. This indicates that our results are useful for a wide range of convex minimization problems on geodesic spaces.

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