CARPATHIAN J. MATH.
Volume **43** (2027), No. 1,
Pages 69-91

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2027.01.05

Continuity and Lipschitz Properties of Approximate Weak Solution Maps via Nonlinear Scalarization

PAKKAPON PREECHASILP[1] AND RABIAN WANGKEEREE[2]

ABSTRACT. This paper examines the stability of approximate weak solution mappings in set optimization problems under the partial order induced by the Minkowski difference. We introduce a scalarization framework that remains continuous when feasible sets and objective mappings are perturbed separately and show that the scalarization mapping is Hölder continuous. By treating feasible and objective variations independently, the analysis provides sharper stability conditions than approaches that consider joint perturbations. Using these continuity properties, we also establish continuity and Lipschitz-type bounds for approximate solution mappings. These results yield a quantitative framework for studying the stability of approximate weak solutions in set optimization.

1. INTRODUCTION

Set optimization has evolved over the past two decades from a technical extension of vector optimization into a mature modelling paradigm in which a single decision generates a set of attainable outcomes. Its scope now applies to multi-criteria decision making, control systems, differential inclusions, welfare economics, financial risk management, and robust optimization under uncertainty [6,7,11,15,20]. Beyond the classical vector criterion, the set criterion approach has been advanced in many directions (see [16,18,22–24]) and references therein.

A unifying theme in this literature is that meaningful analysis hinges on how sets are compared. In set optimization, two principal criteria are commonly used to compare and evaluate sets: the vector criterion and the set criterion. The vector criterion measures optimality by examining individual elements of the image set, whereas the set criterion determines optimality through minimal elements of the collection of image sets with respect to order relations. For classical developments of set-based order relations, we refer the reader to [22,23]. In recent years, alternative set order relations have emerged, most notably the Minkowski-difference based orders introduced by Karaman et al. [18], which have been the focus of extensive research in set optimization; see, for example, [1,2,4,12,21].

In set optimization, stability usually refers to the behavior of minimal solution sets under perturbations, ensuring continuity, robustness, or convergence of solutions when data or parameters vary. Several approaches have been proposed to study this property. For continuity and semicontinuity, stability conditions were established for parametric problems. Hölder and Lipschitz stability has been derived using nonlinear scalarization and distance-based methods [13,14]. Convergence stability has also been analyzed via Painlevé–Kuratowski techniques, clarifying external and internal robustness of solution sets [3,5].

Recent research has significantly advanced the analysis of stability and continuity in set optimization problems formulated with partial orders inspired by Karaman et al. [18].

Received: 18.09.2025. In revised form: 27.12.2025. Accepted: 14.02.2026
2020 *Mathematics Subject Classification.* 49J53, 49K40, 90C31.
Key words and phrases. *set optimization; nonlinear scalarization; approximate weak solutions; continuity.*
Corresponding author: Pakkapon Preechasilp; pakkapon.p@psru.ac.th

In particular, several contributions have focused on stability properties of exact solution sets, including semicontinuity, convergence, well-posedness, and structural results. The stability of parametric set optimization problems was investigated by Preechasilp and Wangkeeree [25], who provided sufficient conditions guaranteeing upper semicontinuity, lower semicontinuity, and closedness of the corresponding solution mappings. They also examined pointwise well-posedness in set optimization, deriving sufficient conditions and establishing connections between pointwise and global notions of well-posedness through scalarization techniques in [26]. Gupta et al. [10] introduced a new scalarization function associated with the partial set order and examined its fundamental properties. Within this framework, they proposed two notions of well-posedness, demonstrated their equivalence with scalar well-posedness, and derived necessary conditions for these concepts. Karuna and Lalitha [19] investigated minimal solutions of set optimization problems defined by partial orders. They analyzed convergence of solution sets in both decision and image spaces and provided scalarization-based characterizations of minimal and weak minimal elements. They also established semicontinuity properties of parametric solution mappings. Kapoor, Ahmad and Sharma [17] proposed generalized convexity concepts for set-valued maps, analyzed lower-level maps, and established path connectedness of solution sets with applications to robust Nash equilibrium games. Zhou, Feng and Ansari [31] introduced and analyzed two notions of well-posedness in set optimization and derived necessary and sufficient conditions for each. By employing a nonlinear scalarization method, they established the equivalence of these well-posedness concepts and illustrated their applicability with several examples. Zhang and Xu [29] investigated a nonlinear scalarization function constructed from the Minkowski-difference order and examined its key properties. They proved that this scalarization is both continuous and convex, and applied these results to set optimization problems. Their analysis further established path connectedness of solution sets and continuity of parametric solution mappings without imposing monotonicity conditions on the objective mapping.

By contrast, only a limited number of works have investigated stability properties of approximate solutions in set optimization. For instance, Gupta and Srivastava [9] introduced the concept of approximate weak minimal solutions under the Minkowski-difference order, analyzed their stability under perturbations, and developed a framework of Levitin–Polyak well-posedness with sufficient conditions and characterizations. Zhao et al. [30] further extended the study of approximate solutions to cases where the ordering cone has a nonempty quasirelative interior, proposed new classes of nonlinear scalarization functions, and derived optimality conditions for approximate efficient and approximate weakly efficient solutions.

After reviewing the literature, it becomes clear that only a few papers address approximate solution sets, whereas scalarization methods are widely recognized as central tools for analyzing minimal solution sets in set optimization. In this work, we focus on approximate solutions in set optimization. This study is driven by three related motivations.

First, we extend the continuity results for nonlinear scalarization functions recently obtained by Zhang and Xu [29] to problems with perturbed feasible sets and objective mappings. This extension is carried out by employing the Hausdorff-type distance technique developed by Han [13], which provides a metric framework for analyzing continuity and stability of solution set mappings.

Second, unlike most prior works, we analyze perturbations of the feasible set and objective mapping separately, leading to sharper stability criteria. This separation yields a sharper understanding of stability behavior and reveals continuity properties that remain hidden when both sources of perturbations are considered together.

Third, to the best of our knowledge, no quantitative stability results have yet been obtained for approximate solution mappings under the partial order relation. The literature has largely concentrated on qualitative notions of stability, such as semicontinuity, set convergence, or well-posedness. By establishing Lipschitz-type properties, we provide the first quantitative framework for measuring the sensitivity of approximate solution maps in this setting.

The structure of the paper is as follows. Section 2 introduces the basic notation, essential definitions, and preliminary results concerning cones, the Minkowski difference, partial orders, and nonlinear scalarization. In Section 3, we investigate continuity and Hölder properties of the scalarization function and present metric-type gap functions that serve as tools for stability analysis. Section 4 focuses on semicontinuity, where we establish lower and upper semicontinuity of approximate weak solution mappings for parametric set optimization problems. In Section 5, we derive Lipschitz continuity results for approximate solution mappings under appropriate assumptions.

2. Preliminaries

We collect basic definitions used throughout the paper. Unless stated otherwise, let X and Y be real normed linear spaces. A subset $C \subset Y$ is called a cone if $\lambda x \in C$ for every $x \in C$ and every $\lambda > 0$. We endow Y with the partial order induced by a fixed convex, pointed cone $C \subset Y$ with nonempty interior (in particular, $0_Y \in C$). The family of nonempty subsets of Y is denoted by $\mathcal{P}_0(Y)$.

For $A \in \mathcal{P}_0(Y)$ and $x \in Y$, we write $\operatorname{int}(A)$ for the (topological) interior of A, $\operatorname{cl}(A)$ for its closure, and

$$\mathbb{B}_Y(x, \varepsilon) := \{y \in Y : \ \|y - x\| < \varepsilon\}$$

for the open ball of radius $\varepsilon > 0$ centered at x; the unit ball is $\mathbb{B}_Y := \mathbb{B}_Y(0_Y, 1)$.

Lemma 2.1. *[28] Let Y be a normed linear space and let $A \subset Y$ be convex with $\operatorname{int}(A) \neq \varnothing$. Then*

$$\operatorname{cl}(\operatorname{int}(A)) \ = \ \operatorname{cl}(A).$$

For nonempty sets $A_1, A_2 \subset Y$, define the Minkowski (Pontryagin) difference

$$A \dot{-} B := \{y \in Y : \ y + B \subset A\}.$$

Proposition 2.1. *Let $A, B \in \mathcal{P}_0(Y)$ and $k \in Y$. The following statements are true:*

(i) $(k + A) \dot{-} B = k + (A \dot{-} B)$.
(ii) $A \dot{-} (k + B) = -k + (A \dot{-} B)$.
(iii) If A is a bounded set, then $A \dot{-} A = \{0_Y\}$.
(iv) If A is closed, then $A \dot{-} B$ is also closed.
(v) If A is compact, then $A \dot{-} B$ is also compact.

Given C as above, the partial set order $\preceq_C^{m_2}$ is

$$A \preceq_C^{m_2} B \quad \Longleftrightarrow \quad (A \dot{-} B) \cap (-C) \neq \emptyset,$$

and

$$A \prec_C^{m_2} B \quad \Longleftrightarrow \quad (A \dot{-} B) \cap \operatorname{int}(-C) \neq \emptyset.$$

Fix $e \in \operatorname{int}(C)$. For given sets $A, B \in X$, the scalarization function associated with A and B is

$$I_e^{m_2}(A, B) := \inf\{t \in \mathbb{R} : A \preceq_C^{m_2} te + B\}.$$

The following Proposition can be found in [18].

Proposition 2.2. *Let $r \in \mathbb{R}$ and $A, B \in \mathcal{P}_0(Y)$. Then the following statements are true:*

(i) $I_e^{m_2}(A, B) < r$ if and only if $A \prec_C^{m_2} re + B$.

(ii) If $B \dot{-} A$ is compact, then $I_e^{m_2}(A,B) = r$ if and only if $A \preceq^{m_2}_{\mathrm{cl}(C)} re + B$ and $A \not\preceq^{m_2}_{C} (r-\varepsilon)e + B$ for all $\varepsilon > 0$.

(iii) If $A \dot{-} B$ is compact, then $I_e^{m_2}(A,B) \le r$ if and only if $A \preceq^{m_2}_{C} re + B$.

(iv) If A is compact, then

$$I_e^{m_2}(A,B) = \min\big\{\, t \in \mathbb{R} \;:\; A \preceq^{m_2}_{C} te + B \,\big\}.$$

Proposition 2.3 ([29]). *Let $A, B \in \mathcal{P}_0(Y)$ and let $\delta \ge 0$. Then*

$$I_e^{m_2}(\delta A, \delta B) \;=\; \delta\, I_e^{m_2}(A,B),$$

where $\delta A := \{\delta a :\; a \in A\}$ and $\delta B := \{\delta b :\; b \in B\}$.

Consider a nonempty feasible set $K \subseteq X$ and a set–valued mapping $F : K \rightrightarrows Y$. The associated set optimization problem with respect to the order $\preceq^{m_2}_{C}$ (abbreviated as (m_2-SOP)) is

(m_2-SOP) $\qquad \min\limits_{C} F(x) \qquad$ subject to $x \in K$.

Definition 2.1 ([18]). *An element $x_0 \in K$ is called*

(i) a minimal solution *of (m_2-SOP) if there is no $x \in K$ with $F(x) \neq F(x_0)$ satisfying $F(x) \preceq^{m_2}_{C} F(x_0)$;*

(ii) a weak minimal solution *of (m_2-SOP) if there is no $x \in K$ satisfying $F(x) \prec^{m_2}_{C} F(x_0)$.*

We denote by $E^{m_2}(F,K)$ the set of all minimal solutions of (m_2-SOP), and by $E^{w,m_2}(F,K)$ the set of all weak minimal solutions. The following recalls the concept of approximate minimal solutions, including ε-minimal and weak ε-minimal solutions of the (m_2–SOP).

Definition 2.2 ([9]). *An element $x_0 \in K$ is called*

(i) an ε–minimal solution *of (m_2-SOP) if there is no $x \in K$ with $F(x) \neq F(x_0)$ satisfying $F(x) + \varepsilon e \preceq^{m_2}_{C} F(x_0)$;*

(ii) a weak ε–minimal solution *of (m_2-SOP) if there is no $x \in K$ satisfying $F(x) + \varepsilon e \prec^{m_2}_{C} F(x_0)$.*

Let $E_\varepsilon^{m_2}(F,K)$ denote the set of all ε–minimal solutions of (m_2-SOP), and $E_\varepsilon^{w,m_2}(F,K)$ the set of all weak ε–minimal solutions.

Remark 2.1. *If $\varepsilon = 0$, then $E_\varepsilon^{m_2}(F,K) = E^{m_2}(F,K)$ and $E_\varepsilon^{w,m_2}(F,K) = E^{w,m_2}(F,K)$.*

The following proposition collects three fundamental properties that will be employed throughout: monotonicity of the ε–minimal solution set in ε, a scalarization test for membership, and a simple sufficient condition for ε–minimality.

Proposition 2.4. *(i) If $0 \le \varepsilon_1 \le \varepsilon_2$ and $F(x)$ has compact values on K, then*

$$E_{\varepsilon_1}^{w,m_2}(F,K) \subseteq E_{\varepsilon_2}^{w,m_2}(F,K).$$

(ii) $x_0 \in E_\varepsilon^{w,m_2}(F,K)$ iff $I_e^{m_2}(F(y),F(x_0)) \ge -\varepsilon$ for all $y \in K$.

(iii) $I_e^{m_2}\big(F(y),F(x)\big) > -\varepsilon$ for all $y \in K \Longrightarrow x \in E_\varepsilon^{m_2}(F,K)$.

Proof. (1) Let $0 \le \varepsilon_1 \le \varepsilon_2$ and suppose $x_1 \in E_{\varepsilon_1}^{w,m_2}(F,K)$. We show $x_1 \in E_{\varepsilon_2}^{w,m_2}(F,K)$. Assume, to the contrary, that $x_1 \notin E_{\varepsilon_2}^{w,m_2}(F,K)$. Then there exists $x_2 \in K$ such that

$$F(x_2) + \varepsilon_2 e \prec^{m_2}_{C} F(x_1).$$

Since $\varepsilon_1 \le \varepsilon_2$, by monotonicity of the m_2-order with respect to translating by te we have

$$F(x_2) + \varepsilon_1 e \preceq^{m_2}_{C} F(x_2) + \varepsilon_2 e \prec^{m_2}_{C} F(x_1),$$

hence $F(x_2) + \varepsilon_1 e \prec^{m_2}_{C} F(x_1)$, which contradicts $x_1 \in E_{\varepsilon_1}^{w,m_2}(F,K)$. Therefore $x_1 \in E_{\varepsilon_2}^{w,m_2}(F,K)$.

(2) ($\Rightarrow$) Suppose $x_0 \in E_\varepsilon^{w,m_2}(F,K)$. By definition 2.2,

$$F(y) + \varepsilon e \not\prec_C^{m_2} F(x_0) \qquad \text{for all } y \in K.$$

This is equivalent to

$$F(y) \not\prec_C^{m_2} F(x_0) - \varepsilon e \qquad \text{for all } y \in K.$$

Proposition 2.2 (i) gives that,

$$I_e^{m_2}(F(y), F(x_0)) \geq -\varepsilon \qquad \text{for all } y \in K.$$

($\Leftarrow$) Conversely, assume

$$I_e^{m_2}(F(y), F(x_0)) \geq -\varepsilon \qquad \text{for all } y \in K.$$

Then by Proposition 2.2 (i), there is no $y \in K$ with $F(y) + \varepsilon e \prec_C^{m_2} F(x_0)$. Therefore $x_0 \in E_\varepsilon^{w,m_2}(F,K)$.

(3) It immediately follows from Proposition 2.2 (iii).

□

The next proposition states that this nesting is continuous from the right: intersecting all tolerances strictly above a given level recovers the set at that level, a fact we will later use to pass from approximate to exact weak minimality as the tolerance vanishes.

Proposition 2.5. *For any $\varepsilon_0 \geq 0$,*

$$\bigcap_{\varepsilon > \varepsilon_0} E_\varepsilon^{w,m_2}(F,K) \;=\; E_{\varepsilon_0}^{w,m_2}(F,K).$$

Proof. By the scalar characterization (Proposition 2.4 (ii)),

$$E_\varepsilon^{w,m_2}(F,K) = \Big\{ x \in K : \; I_e^{m_2}(F(y), F(x)) \geq -\varepsilon \;\; \forall\, y \in K \Big\}.$$

Define the gauge

$$\psi(x) := \sup_{y \in K} \big(- I_e^{m_2}(F(y), F(x)) \big) \in [-\infty, \infty].$$

Then for every $\varepsilon \geq 0$,

$$E_\varepsilon^{w,m_2}(F,K) = \{ x \in K : \; \psi(x) \leq \varepsilon \}.$$

We claim that for any function $\psi : K \to \overline{\mathbb{R}}$ and any $\varepsilon_0 \in \mathbb{R}$,

$$\bigcap_{\varepsilon > \varepsilon_0} \{ x \in K : \psi(x) \leq \varepsilon \} \;=\; \{ x \in K : \psi(x) \leq \inf_{\varepsilon > \varepsilon_0} \varepsilon \}. \tag{2.1}$$

Indeed, write $S_\varepsilon := \{ x : \psi(x) \leq \varepsilon \}$ and $I := (\varepsilon_0, \infty)$.

Let $x \in \bigcap_{\varepsilon \in I} S_\varepsilon$. Then, by definition,

$$\psi(x) \leq \varepsilon \quad \text{for all } \varepsilon \in I.$$

Hence, $\psi(x)$ is a lower bound of I. Consequently,

$$\psi(x) \leq \inf I = \inf_{\varepsilon > \varepsilon_0} \varepsilon,$$

which shows that x belongs to the right-hand side of (2.1).

Conversely, assume that

$$\psi(x) \leq \inf_{\varepsilon > \varepsilon_0} \varepsilon.$$

Since $\inf_{\varepsilon > \varepsilon_0} \varepsilon \leq \varepsilon$ for every $\varepsilon \in I$, it follows that $\psi(x) \leq \varepsilon$ for all $\varepsilon \in I$. Thus, x belongs to $\bigcap_{\varepsilon \in I} S_\varepsilon$.

Thus (2.1) holds. Because $\inf_{\varepsilon>\varepsilon_0}\varepsilon=\varepsilon_0$, we obtain

$$\bigcap_{\varepsilon>\varepsilon_0}\{x:\psi(x)\le\varepsilon\}=\{x:\psi(x)\le\varepsilon_0\}.$$

Hence,

$$\bigcap_{\varepsilon>\varepsilon_0}E_\varepsilon^{w,m_2}(F,K)=\bigcap_{\varepsilon>\varepsilon_0}\{x:\psi(x)\le\varepsilon\}=\{x:\psi(x)\le\varepsilon_0\}=E_{\varepsilon_0}^{w,m_2}(F,K).$$

□

Corollary 2.1. *For any $\varepsilon_0\ge 0$,*

$$\bigcap_{\varepsilon>0}E_\varepsilon^{w,m_2}(F,K)=E^{w,m_2}(F,K).$$

Proof. Taking $\varepsilon_0=0$ in Proposition 2.5 yields the conclusion. □

The following are definitions of continuity for a set-valued mapping $\Gamma:X\rightrightarrows Y$ at a point $x_0\in X$, which can be found in [8].

Definition 2.3. *(1)* Hausdorff upper semicontinuous *(H-usc) at x_0 if, for every neighborhood V of 0 in Y, there exists a neighborhood $U(x_0)$ of x_0 such that for all $x\in U(x_0)$,*

$$\Gamma(x)\ \subset\ \Gamma(x_0)+V.$$

(2) Upper semicontinuous *(usc) at x_0 if, for every neighborhood V of $\Gamma(x_0)$, there exists a neighborhood $U(x_0)$ of x_0 such that for all $x\in U(x_0)$, $\Gamma(x)\subset V$.*

(3) Lower semicontinuous *(lsc) at x_0 if, for every $y\in\Gamma(x_0)$ and every neighborhood V of y, there exists a neighborhood $U(x_0)$ of x_0 such that for all $x\in U(x_0)$, $\Gamma(x)\cap V\ne\varnothing$.*

We say that Γ is H-usc, usc, or lsc on a set $D\subset X$ if it has the corresponding property at each $x\in D$. Moreover, Γ is (Hausdorff) continuous *on D if it is both usc and lsc on D.*

Definition 2.4. *Let (X,d) be a metric space and let $D,E\subset X$ be nonempty. The Hausdorff distance between D and E is*

$$\mathcal{H}(D,E)\ :=\ \max\{\sigma(D,E),\,\sigma(E,D)\},$$

where

$$\sigma(D,E)\ :=\ \sup_{d\in D}\operatorname{dist}(d,E),\qquad \operatorname{dist}(d,E)\ :=\ \inf_{e\in E}d(d,e).$$

Definition 2.5. *A set-valued mapping $\Gamma:X\rightrightarrows Y$ is called q,θ–Hölder continuous at $x_0\in X$ if there exists a neighborhood U of x_0 such that*

$$\Gamma(x_2)\subset\Gamma(x_1)+q\|u_1-u_2\|^\theta\mathbb{B}_Y,\quad\forall x_1,x_2\in U(x_0).$$

We say that Γ is q–Lipschitz at x_0 when $\theta=1$.

Remark 2.2. *For set-valued maps with nonempty compact values, the inclusion condition*

$$\Gamma(x_2)\subset\Gamma(x_1)+q\|x_1-x_2\|^\theta B_Y$$

is equivalent to the Hausdorff–Hölder estimate

$$H(\Gamma(x_1),\Gamma(x_2))\le q\|x_1-x_2\|^\theta.$$

Definition 2.6. *Let D be a nonempty convex subset of X. A set-valued mapping $\Gamma:X\rightrightarrows Y$ is called*

(i) m_2 C–convex on D *[26] if, for any $x_1,x_2\in D$ and any $\alpha\in[0,1]$, one has*

$$\Gamma\big(\alpha x_1+(1-\alpha)x_2\big)\ \preceq_C^{m_2}\ \alpha\,\Gamma(x_1)+(1-\alpha)\,\Gamma(x_2).$$

(ii) m_2–properly C–quasiconvex on D *[29] if, for any* $x_1, x_2 \in D$ *and any* $\alpha \in [0,1]$, *one has*

$$\text{either } \Gamma\big(\alpha x_1 + (1-\alpha)x_2\big) \preceq_C^{m_2} \Gamma(x_1) \quad \text{or} \quad \Gamma\big(\alpha x_1 + (1-\alpha)x_2\big) \preceq_C^{m_2} \Gamma(x_2).$$

When Γ is a scalar-valued mapping, property (i) automatically implies property (ii). However, this implication does not extend to the set-valued case. The following example shows that, for set-valued mappings, m_2 C–convexity does not necessarily imply m_2–proper C–quasiconvexity.

Example 2.1. *Let* $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $D = [0,1]$, *and define the set–valued map*

$$\Gamma(x) := \{(x,\, 1-x)\} \qquad (x \in D).$$

We will show that Γ *is* m_2 C*-convex, but not* m_2*-properly* C*–quasiconvex. For any* $x_1, x_2 \in D$ *and* $\alpha \in [0,1]$,

$$\Gamma\big(\alpha x_1 + (1-\alpha)x_2\big) = \{(\alpha x_1 + (1-\alpha)x_2,\; 1 - \alpha x_1 - (1-\alpha)x_2)\} = \alpha\Gamma(x_1) + (1-\alpha)\Gamma(x_2),$$

hence $\Gamma(\alpha x_1 + (1-\alpha)x_2) \preceq_C^{m_2} \alpha\Gamma(x_1) + (1-\alpha)\Gamma(x_2)$.

On the other hand, take $x_1 = 0$, $x_2 = 1$, *and* $\alpha = \frac{1}{2}$. *Then*

$$\Gamma(0) = \{(0,1)\}, \quad \Gamma(1) = \{(1,0)\}, \quad \Gamma\big(\tfrac{1}{2}\big) = \{(\tfrac{1}{2}, \tfrac{1}{2})\}.$$

Hence $(\frac{1}{2}, \frac{1}{2}) \not\preceq_C^{m_2} (0,1)$ *since* $(0,1) - (\frac{1}{2}, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \notin C$, *and* $(\frac{1}{2}, \frac{1}{2}) \not\preceq_C^{m_2} (1,0)$ *since* $(1,0) - (\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, -\frac{1}{2}) \notin C$. *Therefore,*

$$\Gamma\big(\tfrac{1}{2}\big) \not\preceq_C^{m_2} \Gamma(0) \quad \text{and} \quad \Gamma\big(\tfrac{1}{2}\big) \not\preceq_C^{m_2} \Gamma(1),$$

m_2*-proper* C*–quasiconvexity is violated.*

Proposition 2.6. *[29] Let* $A \subset Y$ *be nonempty and compact, and let* $\Gamma : D \subset X \rightrightarrows Y$ *have nonempty compact values.*

(i) If $\Gamma(\cdot)$ *is* m_2*–properly* C*–quasiconvex on* D, *then* $I_e^{m_2}(A, \cdot)$ *is properly quasiconcave on* D, *that is, for any* $x_1, x_2 \in D$ *and for any* $\alpha \in [0,1]$, *one has*

$$\text{either } I_e^{m_2}\big(A,\, \alpha\Gamma(x_1) + (1-\alpha)\Gamma(x_2)\big) \;\geq\; I_e^{m_2}(\mu_0, \Gamma(x_1))$$

or

$$I_e^{m_2}\big(A,\, \alpha\Gamma(x_1) + (1-\alpha)\Gamma(x_2)\big) \;\geq\; I_e^{m_2}(\mu_0, \Gamma(x_2)).$$

The next lemma establishes the concavity of the scalarization whenever the underlying set-valued mapping is m_2 C-convex.

Lemma 2.2. *Let* $A \subset Y$ *be nonempty and compact, and let* $\Gamma : X \rightrightarrows Y$ *have nonempty compact values. Assume that* $\Gamma(\cdot)$ *is* m_2 C*–convex on* D. *Then the scalarization* $I_e^{m_2}\big(A, \Gamma(\cdot)\big)$ *is concave in the sense that, for all* $x_1, x_2 \in D$ *and* $\alpha \in [0,1]$,

$$I_e^{m_2}\big(A, \Gamma(\alpha x_1 + (1-\alpha)x_2)\big) \;\geq\; \alpha\, I_e^{m_2}(A, \Gamma(x_1)) + (1-\alpha)\, I_e^{m_2}(A, \Gamma(x_2)).$$

Proof. Fix $x_1, x_2 \in D$ and $\alpha \in [0,1]$, and set $\Gamma_i := \Gamma(x_i)$ $(i = 1, 2)$, and $\Gamma_\alpha := \Gamma(\alpha x_1 + (1-\alpha)x_2)$.

By the m_2 C–convexity of $\Gamma(\cdot)$ we have

$$\Gamma_\alpha \preceq_C^{m_2} \alpha\Gamma_1 + (1-\alpha)\Gamma_2.$$

Since $I_e^{m_2}(A, \cdot)$ is m_2–decreasing in its second argument,

$$I_e^{m_2}(A, \Gamma_\alpha) \;\geq\; I_e^{m_2}\big(A, \alpha\Gamma_1 + (1-\alpha)\Gamma_2\big). \tag{2.2}$$

We know that

$$\begin{aligned}\{t \in \mathbb{R} : A \preceq_C^{m_2} te + \big(\alpha\Gamma_1 + (1-\alpha)\Gamma_2\big)\}& \\ \subset \{t \in \mathbb{R} : \alpha A \preceq_C^{m_2} te + \alpha\Gamma_1\} + \{t \in \mathbb{R} : (1-\alpha)A \preceq_C^{m_2} te + (1-\alpha)\Gamma_2\},&\end{aligned}$$

thus

$$\inf\{t\in\mathbb{R}: A\preceq_C^{m_2} te+\big(\alpha\Gamma_1+(1-\alpha)\Gamma_2\big)\}$$
$$\geq \inf\{t\in\mathbb{R}: \alpha A\preceq_C^{m_2} te+\alpha\Gamma_1\}+\inf\{t\in\mathbb{R}: (1-\alpha)A\preceq_C^{m_2} te+(1-\alpha)\Gamma_2\}.$$

This implies

$$I_e^{m_2}\big(A,\alpha\Gamma_1+(1-\alpha)\Gamma_2\big)\geq I_e^{m_2}(\alpha A,\alpha\Gamma_1)+I_e^{m_2}((1-\alpha)A,(1-\alpha)\Gamma_2).$$

Applying Proposition 2.3 together with (2.2), we obtain

$$I_e^{m_2}\big(A,\Gamma(\alpha x_1+(1-\alpha)x_2)\big)\geq \alpha I_e^{m_2}(A,\Gamma_1)+(1-\alpha)I_e^{m_2}(A,\Gamma_2).$$

□

Remark 2.3. *Lemma 2.2 establishes that if Γ is m_2 C-convex on D, then the scalarization $I_e^{m_2}(A,\Gamma(\cdot))$ is concave on D. Proposition 2.6 asserts that if Γ is m_2–properly C–quasiconvex, then $I_e^{m_2}(A,\Gamma(\cdot))$ is properly quasiconcave. Example 2.1 demonstrates that, for set-valued mappings, m_2 C–convexity does not necessarily imply m_2–proper C–quasiconvexity. By contrast, in the scalar setting the concavity conclusion of Lemma 2.2 automatically entails the proper quasiconcavity property of Proposition 2.6.*

3. Continuity and Hölder properties of the scalarizing map

In this subsection we introduce a scalar function built from the nonlinear scalarization $I_e^{m_2}(\cdot,\cdot)$. Our approach is inspired by the work of Han [13, 14], who investigated Hausdorff-type distances and scalarization techniques for set-valued mappings. After introducing a new scalarization for compact set–valued mappings, we proceed to analyze its regularity properties. In particular, we establish both continuity and Hölder continuity of the scalarization function. These results provide the foundation for subsequent stability analysis of approximate solution mappings in set optimization.

Let T_1,T_2 be normed vector spaces and let $A:T_1\rightrightarrows Y$ and $B:T_2\rightrightarrows Y$ be set–valued maps. Define $\xi:T_1\times T_2\to\mathbb{R}\cup\{\pm\infty\}$ by

$$\xi(x,y):=I_e^{m_2}\big(A(x),B(y)\big)=\inf\big\{\,t\in\mathbb{R}:\ A(x)\preceq_C^{m_2} te+B(y)\,\big\},\qquad (x,y)\in T_1\times T_2.$$

We assume that, for every $x\in T_1$ and $y\in T_2$, $A(x)$ and $B(y)$ are nonempty with compact values and that $A(x)\dot{-}B(y)\neq\varnothing$. Under these standing hypotheses (Proposition 2.2),

$$-\infty<\xi(x,y)<\infty,\qquad \forall\,(x,y)\in T_1\times T_2.$$

Proposition 3.7 ([29, Proposition 10]). *Assume that $A(x)\dot{-}B(y)\neq\varnothing$ for every $(x,y)\in T_1\times T_2$. If $A(\cdot)$ and $B(\cdot)$ are continuous with nonempty compact values, then $\xi(\cdot,\cdot)$ is continuous on $T_1\times T_2$.*

We now establish several basic properties of the scalarization $I_e^{m_2}(\cdot,\cdot)$, which serve as the foundation for defining a metric between set-valued mappings.

Proposition 3.8. *Let $A,M,B\subset Y$ be nonempty compact sets with $A\dot{-}M\neq\varnothing$, $M\dot{-}B\neq\varnothing$ and $A\dot{-}B\neq\varnothing$. Then*

$$I_e^{m_2}(A,B)\ \leq\ I_e^{m_2}(A,M)\ +\ I_e^{m_2}(M,B).$$

Proof. Set $\alpha:=I_e^{m_2}(A,M)$ and $\beta:=I_e^{m_2}(M,B)$. By Proposition 2.2 (iii),

$$A\preceq_C^{m_2}(\alpha e+M)\qquad\text{and}\qquad M\preceq_C^{m_2}(\beta e+B).$$

Thus there exist $c_1,c_2\in -C$ with

$$(c_1+\alpha e)+M\subset A\qquad\text{and}\qquad (c_2+\beta e)+B\subset M.$$

Consequently,

$$(c_1+c_2)+(\alpha+\beta)e+B=(c_1+\alpha e)+\big((c_2+\beta e)+B\big)\subset(c_1+\alpha e)+M\subset A.$$

Because C is a cone, $-C$ is closed under addition, hence $c_1+c_2 \in -C$. Therefore $A \preceq_C^{m_2} (\alpha+\beta)e+B$. Proposition 2.2 (iii) yields that $I_e^{m_2}(A,B) \le \alpha+\beta = I_e^{m_2}(A,M)+I_e^{m_2}(M,B)$. □

Corollary 3.2. *Let $A, B \subset Y$ be nonempty compact sets with $A \dot{-} B \neq \varnothing$ and $B \dot{-} A \neq \varnothing$. Then*

$$I_e^{m_2}(A,B) \;+\; I_e^{m_2}(B,A) \;\ge\; 0.$$

Proof. Apply Proposition 3.8 to get $0 = I_e^{m_2}(A,A) \le I_e^{m_2}(A,B) + I_e^{m_2}(B,A)$. □

We now collect some elementary properties of the symmetric gap

$$G_e(A,B) := \max\{I_e^{m_2}(A,B),\, I_e^{m_2}(B,A)\}.$$

On any class of nonempty compact set where all the involved scalarizations are finite, G_e enjoys the following properties.

Proposition 3.9. *Let $A_1, A_2, A_3 \in \mathcal{P}_0(Y)$ be compact sets. Assume that $A_i \dot{-} A_j \neq \varnothing$ for all $i, j \in \{1,2,3\}$, $i \neq j$. Then the following properties hold:*

(a) Symmetry: $G_e(A_1,A_2) = G_e(A_2,A_1)$.
(b) Diagonal: $G_e(A_1,A_1) = 0$.
(c) Nonnegativity: $G_e(A_1,A_2) \ge 0$.
(d) Triangle inequality:

$$G_e(A_1,A_2) \;\le\; G_e(A_1,A_3) + G_e(A_3,A_2).$$

(e) Positive homogeneity: $G_e(\delta A_1, \delta A_2) = \delta\, G_e(A_1,A_2)$ *for all* $\delta \ge 0$.

Proof. Parts (a)–(b) are immediate; (c) follows from Corollary 3.2.

For (d), we apply Proposition 3.8 twice:

$$I_e^{m_2}(A_1,A_2) \;\le\; I_e^{m_2}(A_1,A_3)+I_e^{m_2}(A_3,A_2), \qquad I_e^{m_2}(A_2,A_1) \;\le\; I_e^{m_2}(A_2,A_3)+I_e^{m_2}(A_3,A_1),$$

and then use

$$\max\{x,y\} \;\le\; \max\{a+c,\, b+d\} \;\le\; \max\{a,b\} + \max\{c,d\}.$$

For (e), the conclusion then follows directly from Proposition 2.3. □

The following proposition shows that $I_e^{m_2}(\cdot,\cdot)$ is 1-Lipschitz in each argument with respect to the symmetric gap $G_e(A,B)$.

Proposition 3.10. *Let $A_1, A_2, A_3 \subset Y$ be nonempty compact sets. Assume that $I_e^{m_2}(A_i,A_j)$ is finite for all $i, j \in \{1,2,3\}$ with $i \neq j$. Then*

$$\left|I_e^{m_2}(A_1,A_3) - I_e^{m_2}(A_2,A_3)\right| \;\le\; G_e(A_1,A_2), \tag{3.3}$$

and

$$\left|I_e^{m_2}(A_3,A_1) - I_e^{m_2}(A_3,A_2)\right| \;\le\; G_e(A_1,A_2). \tag{3.4}$$

Proof. By Proposition 3.8,

$$I_e^{m_2}(A_1,A_3) \;\le\; I_e^{m_2}(A_1,A_2) + I_e^{m_2}(A_2,A_3),$$

so

$$I_e^{m_2}(A_1,A_3) - I_e^{m_2}(A_2,A_3) \;\le\; I_e^{m_2}(A_1,A_2).$$

Exchanging A_1 and A_2 gives

$$I_e^{m_2}(A_2,A_3) \;\le\; I_e^{m_2}(A_2,A_1) + I_e^{m_2}(A_1,A_3),$$

hence

$$-\big(I_e^{m_2}(A_1,A_3) - I_e^{m_2}(A_2,A_3)\big) \;\le\; I_e^{m_2}(A_2,A_1).$$

Combining yields

$$\left|I_e^{m_2}(A_1,A_3)-I_e^{m_2}(A_2,A_3)\right| \le \max\{I_e^{m_2}(A_1,A_2),I_e^{m_2}(A_2,A_1)\}=G_e(A_1,A_2),$$

which proves (3.3).

For (3.4), Proposition 3.8 implies

$$I_e^{m_2}(A_3,A_1) \le I_e^{m_2}(A_3,A_2)+I_e^{m_2}(A_2,A_1), \qquad I_e^{m_2}(A_3,A_2) \le I_e^{m_2}(A_3,A_1)+I_e^{m_2}(A_1,A_2).$$

Thus,

$$I_e^{m_2}(A_3,A_1)-I_e^{m_2}(A_3,A_2) \le I_e^{m_2}(A_2,A_1), \qquad I_e^{m_2}(A_3,A_2)-I_e^{m_2}(A_3,A_1) \le I_e^{m_2}(A_1,A_2),$$

and therefore

$$\left|I_e^{m_2}(A_3,A_1)-I_e^{m_2}(A_3,A_2)\right| \le \max\{I_e^{m_2}(A_1,A_2),I_e^{m_2}(A_2,A_1)\}=G_e(A_1,A_2).$$

□

Let Ω_1 and Ω_2 be two normed vector spaces and $T_1, T_2 \subset X$ be nonempty. Define

$$\Upsilon_1 := \Big\{\Phi : T_1 \to 2^Y \;\Big|\; \Phi(x)\neq\varnothing \text{ and bounded } \forall\, x\in T_1\Big\},$$

and

$$\Upsilon_2 := \Big\{\Psi : T_2 \to 2^Y \;\Big|\; \Psi(x)\neq\varnothing \text{ and bounded } \forall\, x\in T_2\Big\}.$$

Let $\Phi_1,\Phi_2\in\Upsilon_1$. The metric $g_1(\Phi_1,\Phi_2)$ is defined by

$$g_1(\Phi_1,\Phi_2) := \sup_{x\in T_1} G_e\big(\Phi_1(x),\Phi_2(x)\big).$$

Similarly, for $\Psi_1,\Psi_2\in\Upsilon_2$, set

$$g_2(\Psi_1,\Psi_2) := \sup_{x\in T_2} G_e\big(\Psi_1(x),\Psi_2(x)\big).$$

We assume that $A_\alpha\in\Upsilon_1$ for any $\alpha\in\Omega_1$ and $B_\beta\in\Upsilon_2$ for any $\beta\in\Omega_2$. The function $\varphi : T_1\times T_2\times\Omega_1\times\Omega_2\to\mathbb{R}$ is defined by

$$\varphi(x,y,\alpha,\beta) := I_e^{m_2}\big(A_\alpha(x),\,B_\beta(y)\big), \qquad \forall (x,y,\alpha,\beta)\in T_1\times T_2\times\Omega_1\times\Omega_2.$$

The next theorem shows that, under mild nonemptiness and continuity assumptions on A_α and B_β (including uniform continuity in the parameters via the metrics g_1,g_2), the scalarization φ is jointly continuous in (x,y,α,β).

Lemma 3.3. *Let T_1,T_2 be nonempty subsets of X, and let Ω_1,Ω_2 be normed parameter spaces. For each $\alpha\in\Omega_1$ and $\beta\in\Omega_2$, let $A_\alpha : T_1\rightrightarrows Y, B_\beta : T_2\rightrightarrows Y$ be set–valued maps with nonempty compact values. Assume that*

(i) $A_\alpha(x)\dotplus B_\beta(y)\neq\varnothing$ for all $(x,y,\alpha,\beta)\in T_1\times T_2\times\Omega_1\times\Omega_2$;
(ii) for any $(\alpha,\beta)\in\Omega_1\times\Omega_2$, the mappings $A_\alpha(\cdot), B_\beta(\cdot)$ are continuous;
(iii) $g_1(A_\alpha,A_{\alpha_0})\to 0$ as $\alpha\to\alpha_0$ and $g_2(B_\beta,B_{\beta_0})\to 0$ as $\beta\to\beta_0$.

Then φ is continuous at every point $(x_0,y_0,\alpha_0,\beta_0)\in T_1\times T_2\times\Omega_1\times\Omega_2$.

Proof. Let $\varepsilon>0$. For any fixed (α_0,β_0), it follows from (ii) and Proposition 3.7 that the map

$$(x,y)\longmapsto I_e^{m_2}\big(A_{\alpha_0}(x),B_{\beta_0}(y)\big)$$

is continuous at (x_0,y_0). Then there exists $\delta_{xy}>0$ such that

$$\|x-x_0\|<\delta_{xy},\ \|y-y_0\|<\delta_{xy} \ \Rightarrow\ \left|\varphi(x,y,\alpha_0,\beta_0)-\varphi(x_0,y_0,\alpha_0,\beta_0)\right|<\tfrac{\varepsilon}{3}. \tag{3.5}$$

By (iii), there exists $\delta_\omega>0$ such that

$$\|\alpha-\alpha_0\|<\delta_\alpha \Rightarrow g_1(A_\alpha,A_{\alpha_0})<\frac{\varepsilon}{3}.$$

Using Proposition 3.10,

$$\begin{aligned}\big|\varphi(x,y,\alpha,\beta)-\varphi(x,y,\alpha_0,\beta)\big| &= \big|I_e^{m_2}(A_\alpha(x),B_\alpha(y))-I_e^{m_2}(A_{\alpha_0}(x),B_\alpha(y))\big|\\ &\le G_e(A_\alpha(x),A_{\alpha_0}(x))\\ &\le g_1(A_\alpha,A_{\alpha_0})\le\frac{\varepsilon}{3}.\end{aligned}\tag{3.6}$$

Likewise, there exists $\delta_\beta>0$ such that $\|\beta-\beta_0\|<\delta_\beta\Rightarrow g_2(B_\beta,B_{\beta_0})<\frac{\varepsilon}{3}$, and again by Proposition 3.10,

$$\big|\varphi(x,y,\alpha_0,\beta)-\varphi(x,y,\alpha_0,\beta_0)\big|\le G_e(B_\beta(x),B_{\beta_0}(x))\le g_2(B_\beta,B_{\beta_0})\le\frac{\varepsilon}{3}.\tag{3.7}$$

Set $\delta:=\min\{\delta_{xy},\delta_\alpha,\delta_\beta\}$. If $\|(x,y,\alpha,\beta)-(x_0,y_0,\alpha_0,\beta_0)\|<\delta$, then by (3.6), (3.7), and (3.5) give that

$$\begin{aligned}\big|\varphi(x,y,\alpha,\beta)-\varphi(x_0,y_0,\alpha_0,\beta_0)\big| &\le \big|\varphi(x,y,\alpha,\beta)-\varphi(x,y,\alpha_0,\beta)\big|\\ &\quad+\big|\varphi(x,y,\alpha_0,\beta)-\varphi(x,y,\alpha_0,\beta_0)\big|\\ &\quad+\big|\varphi(x,y,\alpha_0,\beta_0)-\varphi(x_0,y_0,\alpha_0,\beta_0)\big|\\ &<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\;=\;\varepsilon.\end{aligned}$$

Thus φ is continuous at $(x_0,y_0,\alpha_0,\beta_0)$. □

Remark 3.4. *Lemma 3.3 establishes the continuity of the parametric scalarization $\varphi(x,y,\alpha,\beta)$ by combining the continuity of the set–valued maps $A_\alpha(\cdot)$ and $B_\beta(\cdot)$ with the gap function estimates in Propositions 3.9 and 3.10. This result can be viewed as a parametric extension of Proposition 10 in [29], where continuity of the scalarization was proved for fixed mappings $A(\cdot)$ and $B(\cdot)$ without perturbations.*

We now work under the following local hypotheses, which ensure finiteness of the scalarization and Hölder property of the data in the state variables and parameters near $(x_0,y_0,\alpha_0,\beta_0)$.

(H1) $(A_\alpha(\cdot)\dot{-}B_\beta(\cdot))\neq\varnothing$ for all $(\alpha,\beta)\in U(\alpha_0)\times U(\beta_0)$.

(H2) There exist $T\ge0$ and $\theta>0$ such that, for any $\alpha\in U(\alpha_0)$ and any $x_1,x_2\in U(x_0)$,

$$G_e\big(A_\alpha(x_1),A_\alpha(x_2)\big)\le T\|x_1-x_2\|^\theta.$$

(H3) There exist $L\ge0$ and $\nu>0$ such that, for any $\beta\in U(\beta_0)$ and any $y_1,y_2\in U(y_0)$,

$$G_e\big(B_\beta(y_1),B_\beta(y_2)\big)\le L\|y_1-y_2\|^\nu.$$

(H4) There exist $M\ge0$ and $\gamma>0$ such that, for any $\alpha_1,\alpha_2\in U(\alpha_0)$,

$$g_1(A_{\alpha_1},A_{\alpha_2})\;\le\;M\,\|\alpha_1-\alpha_2\|^\gamma.$$

(H5) There exist $V\ge0$ and $\eta>0$ such that, for any $\beta_1,\beta_2\in U(\beta_0)$,

$$g_2(B_{\beta_1},B_{\beta_2})\;\le\;V\,\|\beta_1-\beta_2\|^\eta.$$

Theorem 3.1. *Let $(x_0,y_0,\alpha_0,\beta_0)\in T_1\times T_2\times\Omega_1\times\Omega_2$ and let $U(x_0)\times U(y_0)\times U(\alpha_0)\times U(\beta_0)$ be a neighborhood of $(x_0,y_0,\alpha_0,\beta_0)$ such that* Assumptions (H1)–(H5) *hold. Then for any $(x_1,y_1,\alpha_1,\beta_1),(x_2,y_2,\alpha_2,\beta_2)\in U(x_0)\times U(y_0)\times U(\alpha_0)\times U(\beta_0)$,*

$$\begin{aligned}&\big|\varphi(x_1,y_1,\alpha_1,\beta_1)-\varphi(x_2,y_2,\alpha_2,\beta_2)\big|\\ &\qquad\le\;T\,\|x_1-x_2\|^\theta\;+\;L\,\|y_1-y_2\|^\nu\;+\;M\|\alpha_1-\alpha_2\|^\gamma\;+\;V\|\beta_1-\beta_2\|^\eta.\end{aligned}$$

Proof. Let $(x_1, y_1, \alpha_1, \beta_1), (x_2, y_2, \alpha_2, \beta_2) \in U(x_0) \times U(y_0) \times U(\alpha_0) \times U(\beta_0)$. We have

$$\begin{aligned}&\big|\varphi(x_1, y_1, \alpha_1, \beta_1) - \varphi(x_2, y_2, \alpha_2, \beta_2)\big| \\ &\leq \big|\varphi(x_1, y_1, \alpha_1, \beta_1) - \varphi(x_2, y_2, \alpha_1, \beta_1)\big| + \big|\varphi(x_2, y_2, \alpha_1, \beta_1) - \varphi(x_2, y_2, \alpha_2, \beta_2)\big|. \end{aligned} \tag{3.8}$$

For any fixed $(\alpha_1, \beta_1) \in U(\alpha_0) \times U(\beta_0)$, Proposition 3.10, (H2) and (H3) give that

$$\begin{aligned}\big|\varphi(x_1, y_1, \alpha_1, \beta_1) - \varphi(x_2, y_2, \alpha_1, \beta_1)\big| &\leq \big|I_e^{m_2}\big(A_{\alpha_1}(x_1), B_{\beta_1}(y_1)\big) - I_e^{m_2}\big(A_{\alpha_1}(x_2), B_{\beta_1}(y_1)\big)\big| \\ &\quad + \big|I_e^{m_2}\big(A_{\alpha_1}(x_2), B_{\beta_1}(y_1)\big) - I_e^{m_2}\big(A_{\alpha_1}(x_2), B_{\beta_1}(y_2)\big)\big| \\ &\leq T\|x_1 - x_2\|^{\theta} + L\|y_1 - y_2\|^{\nu}. \end{aligned} \tag{3.9}$$

It follows from Proposition 3.10, (H4) and (H5) that

$$\begin{aligned}\big|\varphi(x_2, y_2, \alpha_1, \beta_1) - \varphi(x_2, y_2, \alpha_2, \beta_2)\big| &\leq G_e\big(A_{\alpha_1}(x_2), A_{\alpha_2}(x_2)\big) + G_e\big(B_{\beta_1}(y_2), B_{\beta_2}(y_2)\big) \\ &\leq g_1(A_{\alpha_1}, A_{\alpha_2}) + g_2(B_{\beta_1}, B_{\beta_2}) \\ &\leq M\|\omega_1 - \omega_2\|^{\gamma} + V\|\alpha_1 - \alpha_2\|^{\eta}. \end{aligned}$$

Combining this with (3.9) in (3.8) yields the desired estimate. □

Remark 3.5. *Theorem 3.1 establishes a new continuity result for the parametric scalarization map φ associated with (m_2-PSOP). In particular, it shows that φ enjoys Hölder continuity with respect to both state variables and parameters, with exponents $\theta, \nu, \gamma, \eta$ determined by the local hypotheses (H1)–(H5). This extends earlier continuity results for scalarization functions by demonstrating quantitative Hölder-type stability in the framework of the Minkowski-difference order, which has not been previously investigated in the literature.*

Example 3.2. *Let $X = Y = \mathbb{R}$, $C = [0, \infty)$ and $e = 1$. Consider the parameter spaces $\Omega_1 = \Omega_2 = \mathbb{R}$ and define the set–valued maps*

$$A_\alpha(x) := [\, x + \alpha,\ x + \alpha + 1\,], \qquad B_\beta(y) := [\, y + \beta,\ y + \beta + 1\,].$$

Each $A_\alpha(x)$ and $B_\beta(y)$ is a compact interval in $\mathbb{R}$, and the mappings are continuous with respect to both x, y and the parameters α, β.

We now verify the assumptions (H1)–(H5).

(H1). *For any (x, y, α, β),*

$$A_\alpha(x) \dot{-} B_\beta(y) = [\, x + \alpha - y - \beta,\ x + \alpha - y - \beta\,]$$

is a singleton, hence nonempty.

(H2)–(H3). *For $A_\alpha(x_1) = [x_1 + \alpha,\ x_2 + \alpha + 1]$ and $A_\alpha(x_2) = [x_2 + \alpha,\ x_2 + \alpha + 1]$, we have*

$$I_e^{m_2}\big(A_\alpha(x_1), A_\alpha(x_2)\big) = x_1 - x_2,$$

and symmetrically

$$I_e^{m_2}\big(A_\alpha(x_2), A_\alpha(x_1)\big) = x_2 - x_1.$$

Hence $G_e\big(A_\alpha(x_1), A_\alpha(x_2)\big) = |x_1 - x_2|$, and so $G_e\big(B_\beta(y_1), B_\beta(y_2)\big) = |y_1 - y_2|$.

(H4)–(H5). *Similarly, for $A_{\alpha_1}(x) = [x + \alpha_1,\ x + \alpha_1 + 1]$ and $A_{\alpha_2}(x) = [x + \alpha_2,\ x + \alpha_2 + 1]$ one has*

$$G_e(A_{\alpha_1}, A_{\alpha_2}) = |\alpha_1 - \alpha_2|.$$

Therefore $g(A_{\alpha_1}, A_{\alpha_2}) = |\alpha_1 - \alpha_2|$, and so $g_2(B_{\beta_1}, B_{\beta_2}) = |\beta_1 - \beta_2|$.

By Theorem 3.1, we obtain for any $(x_1, y_1, \alpha_1, \beta_1), (x_2, y_2, \alpha_2, \beta_2)$ in a neighborhood of a fixed $(x_0, y_0, \alpha_0, \beta_0)$,

$$\big|\varphi(x_1, y_1, \alpha_1, \beta_1) - \varphi(x_2, y_2, \alpha_2, \beta_2)\big| \ \leq\ |x_1 - x_2| + |y_1 - y_2| + |\alpha_1 - \alpha_2| + |\beta_1 - \beta_2|.$$

4. Continuity of approximate solution maps

Let (κ, μ) be a parameter varying in a subset $\Lambda \times M$ of a normed space. For each (κ, μ), consider a feasible set $K(\kappa) \subseteq X$ and a set–valued mapping $F(\cdot, \mu) : K(\kappa) \rightrightarrows Y$. The corresponding parametric set optimization problem with respect to the order $\preceq_C^{m_2}$ (abbreviated as $(m_2$-PSOP)) is

(m_2-PSOP) $\qquad \min_C F(x, \mu) \qquad$ subject to $x \in K(\kappa)$.

For each $(\varepsilon, \kappa, \mu) \in [0, \infty) \times \Lambda \times M$, let $S(\varepsilon, \kappa, \mu)$ and $S^w(\varepsilon, \kappa, \mu)$ denote, respectively, the ε-approximate m_2-minimal solution set and the ε-approximate weak m_2-minimal solution set of $(m_2$-PSOP$)$.

We introduce the function

$$\zeta(x, y, \mu) := I_e^{m_2}\big(F(x, \mu), F(y, \mu)\big).$$

Based on this scalarization ζ, we define

$$S_\zeta^{>}(\varepsilon, \kappa, \mu) := \Big\{x \in K(\kappa) : \ \zeta(y, x, \mu) > -\varepsilon, \ \ \forall\, y \in K(\kappa)\Big\},$$

$$S_\zeta^{w}(\varepsilon, \kappa, \mu) := \Big\{x \in K(\kappa) : \ \zeta(y, x, \mu) \geq -\varepsilon, \ \ \forall\, y \in K(\kappa)\Big\}.$$

By Proposition 2.4 (ii) and (iii), we have

$$S_\zeta^{>}(\varepsilon, \kappa, \mu) \ \subset \ S(\varepsilon, \kappa, \mu) \subset S^w(\varepsilon, \kappa, \mu) \ = \ S_\zeta^{w}(\varepsilon, \kappa, \mu).$$

Throughout, we assume $S_\zeta^{>}(\varepsilon, \kappa, \mu) \neq \varnothing$.

Theorem 4.2. *Fix $e \in \operatorname{int}(C)$. Let $\varepsilon_0 \geq 0$ and $(\kappa_0, \mu_0) \in \Lambda \times M$. Assume that*

(i) $F(y, \mu) \dot{-} F(x, \mu) \neq \emptyset$ for all $x, y \in K(\kappa_0)$ and $\mu \in M$;
(ii) $F(\cdot, \mu_0)$ is continuous with nonempty compact values on $K(\kappa_0) \times \{\mu_0\}$;
(iii) $\lim\limits_{\mu \to \mu_0} g(F_\mu, F_{\mu_0}) = 0$;
(iv) $K(\cdot)$ is continuous at κ_0 with nonempty compact values.

Then $S_\zeta^w(\cdot, \cdot, \cdot)$ is usc at $(\varepsilon_0, \kappa_0, \mu_0)$ and $S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0)$ is compact.

Proof. Suppose, for contradiction, that S_ζ^w is not usc at $(\varepsilon_0, \kappa_0, \mu_0)$. Then there exist a neighborhood U of $S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0)$ and a sequence $(\varepsilon_n, \kappa_n, \mu_n) \to (\varepsilon_0, \kappa_0, \mu_0)$ together with $x_n \in S_\zeta^w(\varepsilon_n, \kappa_n, \mu_n)$ such that $x_n \notin U$ for all n.

By continuity of $K(\cdot)$ at κ_0, $\{x_n\}$ is relatively compact; passing to a subsequence, $x_n \to x_0 \in K(\kappa_0)$.

Suppose $x_0 \notin S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0)$. Then there exists $y_0 \in K(\kappa_0)$ such that

$$\zeta(y_0, x_0, \mu_0) < -\varepsilon_0. \tag{4.10}$$

By the lower semicontinuity of $K(\cdot)$ at κ_0, there is $y_n \in K(\kappa_n)$ with $y_n \to y_0$. Since $x_n \in S_\zeta^w(\varepsilon_n, \kappa_n, \mu_n)$ and $y_n \in K(\kappa_n)$, we have

$$\zeta(y_n, x_n, \mu_n) \ \geq \ -\varepsilon_n \qquad \text{for all } n. \tag{4.11}$$

By (ii) and (iii), together with Lemma 3.3, we obtain

$$\zeta(y_n, x_n, \mu_n) \longrightarrow \zeta(y_0, x_0, \mu_0) \quad \text{and} \quad -\varepsilon_n \longrightarrow -\varepsilon_0.$$

Passing to the limit in (4.11) yields $\zeta(y_0, x_0, \mu_0) \geq -\varepsilon_0$, contradicting (4.10). Hence $x_0 \in S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0)$. Because U is a neighborhood of this set, $x_n \in U$ for all sufficiently large n, contradiction. Therefore $S_\zeta^w(\cdot, \cdot, \cdot)$ is upper semicontinuous at $(\varepsilon_0, \kappa_0, \mu_0)$.

To establish the compactness of $S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0)$, note that it is contained in the compact set $K(\kappa_0)$. Therefore, it remains only to verify that $S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0)$ is closed. Let $x_n \in$

$S_\zeta^w(\varepsilon_0,\kappa_0,\mu_0)$ with $x_n \to x_0$. Since $K(\kappa_0)$ is a closed set, one has $x_0 \in K(\kappa_0)$. Let $y \in K(\kappa_0)$. Then for every $n \in \mathbb{N}$,

$$\zeta(y,x_n,\mu_0) \geq -\varepsilon_0.$$

Using the continuity of $F(\cdot,\mu_0)$ and Lemma 3.3 we obtain

$$\zeta(y,x_n,\mu_0) \to \zeta(y,x_0,\mu_0).$$

Hence $\zeta(y,x_0,\mu_0) \geq -\varepsilon_0$ for every $y \in K(\kappa_0)$, i.e., $x_0 \in S_\zeta^w(\varepsilon_0,\kappa_0,\mu_0)$. Thus $S_\zeta^w(\varepsilon_0,\kappa_0,\mu_0)$ is closed in the compact set $K(\kappa_0)$, hence it is compact. □

The next example shows that assumption (iii) in Theorem 4.3 is indispensable: if $g(F_\mu,F_{\mu_0}) \not\to 0$, the ε–weak solution map need not be usc.

Example 4.3. *Let $X = Y = \mathbb{R}$, $\Lambda = M = \mathbb{R}$, and equip Y with the order cone $C = [0,\infty)$; fix $e = 1 \in \operatorname{int}(C)$. Let the feasible set mapping be constant*

$$K(\kappa) \equiv [0,1].$$

Fix $\delta \in (0,1)$, and define a compact interval–valued mapping $F : X \times M \rightrightarrows Y$ by

$$F(x,\mu) = \begin{cases} \big[-\mu^{-1}x - \delta,\ -\mu^{-1}x\big], & \mu \neq 0, \\ \big[x-\delta,\ x\big], & \mu = 0. \end{cases}$$

With $\mu_0 = 0$, the mapping $F(\cdot,\mu_0)$ is continuous on $K(\kappa_0)$. In addition, it has nonempty compact values on $K(\kappa_0)$. Since the Minkowski difference of two intervals is again an interval, it follows that $F(y,\mu) \dot{-} F(x,\mu) \neq \varnothing$ for all $x,y \in K(\kappa_0)$ and $\mu \in M$. Hence assumption (i)–(ii) *of Theorem 4.2 holds. Because $K(\kappa) \equiv [0,1]$, assumption (iv) holds immediately.*

From definition of F, we have that

$$G_e\big(F(x,\mu),F(x,0)\big) = \max\Big\{(-\mu^{-1}x-\delta)-x,\ (x-\delta)-(-\mu^{-1}x)\Big\} = \big|\mu^{-1}x+x\big| - \delta.$$

Taking the supremum in $x \in [0,1]$ gives

$$g(F_\mu,F_{\mu_0}) = \sup_{x\in[0,1]} G_e\big(F(x,\mu),F(x,0)\big) \ \geq\ \big|\mu^{-1}+1\big| - \delta \ \to \infty \ \text{ as } \ \mu \to 0.$$

Thus assumption (iii) *fails.*

Again, from the definition of F,

$$\zeta(y,x,\mu) = \begin{cases} -\mu^{-1}(y-x) - \delta, & \mu \neq 0, \\ (y-x)-\delta, & \mu = 0. \end{cases}$$

Let $\varepsilon = \delta$. Then

$$\begin{aligned} x \in S_\zeta^w(\delta,\kappa_0,0) &\iff (y-x)-\delta \geq -\delta \ \ \forall\, y \in [0,1] \\ &\iff x = 0. \end{aligned}$$

For $\mu > 0$,

$$\begin{aligned} x \in S_\zeta^w(\delta,\kappa_0,\mu) &\iff -\mu^{-1}(y-x)-\delta \geq -\delta \ \ \forall\, y \in [0,1] \\ &\iff -\mu^{-1}(y-x) \geq 0 \ \ \forall\, y \in [0,1] \\ &\iff x = 1. \end{aligned}$$

Therefore,

$$S_\zeta^w(\delta,\kappa_0,0) = \{0\}, \qquad S_\zeta^w(\delta,\kappa_0,\mu) = \{1\} \quad (\mu > 0).$$

Define $\mu_n := \frac{1}{n}$ *and* $x_n := 1 \in S_\zeta^w(\delta, \kappa_0, \mu_n)$ *for all* n*, so that* $x_n \to 1$*. However,* $x = 1 \notin S_\zeta^w(\delta, \kappa_0, 0) = \{0\}$*. Equivalently, taking the neighborhood* $W = [0, \frac{1}{2})$ *of* $\{0\}$*, one has*

$$S_\zeta^w(\delta, \kappa_0, \mu_n) = \{1\} \not\subset W$$

for sufficiently small $\mu_n > 0$*, contradicting upper semicontinuity.*

Theorem 4.3. *Fix* $e \in \operatorname{int}(C)$*. Let* $\varepsilon_0 \geq 0$ *and* $(\kappa_0, \mu_0) \in \Lambda \times M$*. Assume that*

(i) $F(y, \mu) \dot{-} F(x, \mu) \neq \emptyset$ *for all* $x, y \in K(\kappa_0)$ *and* $\mu \in M$*;*
(ii) $F(\cdot, \mu_0)$ *has nonempty compact values and is continuous on* $K(\kappa_0) \times \{\mu_0\}$*;*
(iii) $\lim\limits_{\mu \to \mu_0} g\,(F_\mu, F_{\mu_0}) = 0$*;*
(iv) $K(\cdot)$ *is continuous at* κ_0 *with nonempty compact values.*

Then $S_\zeta^>(\cdot, \cdot, \cdot)$ *is lsc at* $(\varepsilon_0, \kappa_0, \mu_0)$*.*

Proof. Assume that $S_\zeta^>(\cdot, \cdot, \cdot)$ is not lsc at $(\varepsilon_0, \kappa_0, \mu_0)$. Then there exist $x_0 \in S_\zeta^>(\varepsilon_0, \kappa_0, \mu_0)$, a neighborhood U of x_0 in X, and a sequence $(\varepsilon_n, \kappa_n, \mu_n) \to (\varepsilon_0, \kappa_0, \mu_0)$ such that

$$S_\zeta^>(\varepsilon_n, \kappa_n, \mu_n) \cap U = \varnothing, \qquad \forall n \in \mathbb{N}. \tag{4.12}$$

Since $K(\cdot)$ is continuous at κ_0 with compact values, there exists a sequence $v_n \in K(\kappa_n)$ with $v_n \to x_0$. Since U is a neighborhood of x_0, there exists $N \in \mathbb{N}$ with for $n > N$ we have $v_n \in U$. By (4.12) we must have $v_n \notin S_\zeta^>(\varepsilon_n, \kappa_n, \mu_n)$, hence, by the definition of $S_\zeta^>$, there exists $y_n \in K(\kappa_n)$ such that

$$\zeta(y_n, v_n, \mu_n) \;\leq\; -\varepsilon_n, \qquad \forall\, n > N. \tag{4.13}$$

By the upper semicontinuity of $K(\cdot)$ at κ_0 and the compactness of $K(\kappa_0)$, there exists, if necessary, a subsequence such that $y_n \to y_0$ for some $y_0 \in K(\kappa_0)$. Using the continuity of $F(\cdot, \mu_0)$ on $K(\kappa_0) \times \{\mu_0\}$ at the points (y_0, μ_0) and (x_0, μ_0), the convergence $(y_n, v_n, \kappa_n) \to (y_0, x_0, \kappa_0)$ implies

$$G_e\big(F(y_n, \mu_0), F(y_0, \mu_0)\big) \to 0, \qquad G_e\big(F(v_n, \mu_0), F(x_0, \mu_0)\big) \to 0.$$

Moreover, by (iii),

$$G_e\big(F(z, \mu_n), F(z, \mu_0)\big) \;\leq\; g(F_{\mu_n}, F_{\mu_0}) \to 0 \text{ as } n \to \infty.$$

Applying Lemma 3.3, we obtain

$$\zeta(y_n, v_n, \mu_n) \;\longrightarrow\; \zeta(y_0, x_0, \mu_0).$$

Passing to the limit in (4.13) yields

$$\zeta(y_0, x_0, \mu_0) \;\leq\; -\varepsilon.$$

But $x_0 \in S_\zeta^>(\varepsilon_0, \kappa_0, \mu_0)$, which leads to a contradiction. This shows that $S_\zeta^>(\cdot, \cdot, \cdot)$ is lsc at $(\varepsilon_0, \kappa_0, \mu_0)$. □

Lemma 4.4. *Fix* $e \in \operatorname{int}(C)$*. Let* $\varepsilon_0 \geq 0$ *and* $(\kappa_0, \mu_0) \in \Lambda \times M$*. Assume that*

(i) $F(y, \mu) \dot{-} F(x, \mu) \neq \varnothing$ *for all* $x, y \in K(\kappa_0)$ *and all* $\mu \in M$*.*
(ii) $F(\cdot, \mu_0)$ *has nonempty compact values and is continuous on* $K(\kappa_0) \times \{\mu_0\}$*;*
(iii) $\lim\limits_{\mu \to \mu_0} g(F_\mu, F_{\mu_0}) = 0$*;*
(iv) $F(\cdot, \mu_0)$ *is* m_2*-properly* C*-quasiconvex on* $K(\kappa_0)$*;*
(v) $K(\cdot)$ *is continuous at* κ_0 *with nonempty compact values;*

Then for every $\varepsilon \geq 0$*,*

$$S_\zeta^>(\varepsilon_0, \kappa_0, \mu_0) \;\subset\; S(\varepsilon_0, \kappa_0, \mu_0) \;\subset\; S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0) \quad \textit{and} \quad S_\zeta^w(\varepsilon_0, \kappa_0, \mu_0) \;=\; \operatorname{cl}\big(S_\zeta^>(\varepsilon_0, \kappa_0, \mu_0)\big).$$

Proof. By definition,

$$S^{>}(\varepsilon_0,\kappa_0,\mu_0)=\big\{x\in K(\kappa_0):\ \zeta(y,x,\mu_0)>-\varepsilon,\ \ \forall\, y\in K(\kappa_0)\big\},$$

and Proposition 2.4 (iii), the first inclusion follows immediately. The second inclusion follows directly.

We want to show that $S^w_\zeta(\varepsilon_0,\kappa_0,\mu_0)$ is closed and convex. By Theorem 4.2, $S^w_\zeta(\varepsilon_0,\kappa_0,\mu_0)$ is closed.

Let $x_1,x_2\in S^w_\zeta(\varepsilon_0,\kappa_0,\mu_0)$ and $t\in[0,1]$. Then by (iv) and Proposition 2.6 one has either

$$\zeta(y,tx_1+(1-t)x_2,\mu_0)\geq\zeta(y,x_1,\mu_0)\geq-\varepsilon_0,\quad\forall y\in K(\kappa_0)$$

or

$$\zeta(y,tx_1+(1-t)x_2,\mu_0)\geq\zeta(y,x_2,\mu_0)\geq-\varepsilon_0,\quad\forall y\in K(\kappa_0).$$

This means that $tx_1+(1-t)x_2\in S^w(\varepsilon_0,\kappa_0,\mu_0)$. It implies that $S^w(\varepsilon_0,\kappa_0,\mu_0)$ is convex.

By Lemma 2.1, we have $S^w_\zeta(\varepsilon_0,\kappa_0,\mu_0)=\mathrm{cl}\,(S^{>}_\zeta(\varepsilon_0,\kappa_0,\mu_0))$. □

Theorem 4.4. *Fix $e\in\mathrm{int}(C)$. Let $\varepsilon_0\geq0$ and $(\kappa_0,\mu_0)\in\Lambda\times M$. Assume that*

(i) $F(y,\mu)\dot{-}F(x,\mu)\neq\varnothing$ *for all $x,y\in K(\kappa_0)$ and all $\mu\in M$;*
(ii) $F(\cdot,\mu_0)$ *has nonempty compact values and is continuous on $K(\kappa_0)\times\{\mu_0\}$;*
(iii) $\lim\limits_{\mu\to\mu_0} g(F_\mu,F_{\mu_0})=0$;
(iv) $F(\cdot,\mu_0)$ *is m_2-properly quasiconvex on $K(\kappa_0)$;*
(v) $K(\cdot)$ *is continuous at κ_0 with nonempty compact values.*

Then $S(\cdot,\cdot,\cdot)$ is H-usc at $(\varepsilon_0,\kappa_0,\mu_0)$.

Proof. Assume to the contrary that the conclusion fails. Then there exists an open set $0_X\in U_0$ and a sequence $(\varepsilon_n,\kappa_n,\mu_n)\to(\varepsilon_0,\kappa_0,\mu_0)$ such that

$$x_n\in S(\varepsilon_n,\kappa_n,\mu_n)\ \text{ and }\ x_n\notin S(\varepsilon_0,\kappa_0,\mu_0)+U_0,\qquad\forall n\in\mathbb{N}. \tag{4.14}$$

Since $x_n\in K(\kappa_n)$ and $K(\cdot)$ has compact values and is continuous at κ_0, there is a subsequence, if necessary, with

$$x_n\to x_0\in K(\kappa_0).$$

By Lemma 4.4, we have

$$x_0\in S(\varepsilon,\kappa_0,\mu_0).$$

Again, by Lemma 4.4 we have

$$\mathrm{cl}\,(S(\varepsilon_0,\kappa_0,\mu_0))\ =\ S^w(\varepsilon_0,\kappa_0,\mu_0).$$

In particular, $x_0\in\mathrm{cl}(S(\varepsilon_0,\kappa_0,\mu_0))\subset S(\varepsilon_0,\kappa_0,\mu_0)+U_0$ because U_0 is an open neighborhood of the origin. Since $S(\varepsilon_0,\kappa_0,\mu_0)+U_0$ is open and $x_n\to x_0$, we must have $x_n\in S(\varepsilon_0,\kappa_0,\mu_0)+U_0$ for all sufficiently large n, contradicting (4.14). Hence $S(\cdot,\cdot,\cdot)$ is H-usc at $(\varepsilon_0,\kappa_0,\mu_0)$. □

Theorem 4.5. *Fix $e\in\mathrm{int}(C)$. Let $\varepsilon_0\geq0$ and $(\kappa_0,\mu_0)\in\Lambda\times M$. Assume that*

(i) $F(y,\mu)\dot{-}F(x,\mu)\neq\varnothing$ *for all $x,y\in K(\kappa_0)$ and all $\mu\in M$;*
(ii) $F(\cdot,\mu_0)$ *has nonempty compact values and is continuous on $K(\kappa_0)\times\{\mu_0\}$;*
(iii) $\lim\limits_{\mu\to\mu_0} g\,(F_\mu,F_{\mu_0})=0$;
(iv) $F(\cdot,\mu_0)$ *is m_2-properly quasiconvex on $K(\kappa_0)$;*
(v) $K(\cdot)$ *is continuous at κ_0 with nonempty compact values.*

Then $S(\cdot,\cdot,\cdot)$ and $S^w(\cdot,\cdot,\cdot)$ are lsc at $(\varepsilon_0,\kappa_0,\mu_0)$.

Proof. Let $x_0 \in S(\varepsilon_0, \kappa_0, \mu_0)$ and let $U(x_0)$ be any neighborhood of x_0 in X. By Lemma 4.4, we have

$$S(\varepsilon_0, \kappa_0, \mu_0) = \mathrm{cl}(S_\zeta^>(\varepsilon_0, \kappa_0, \mu_0)) = S^w(\varepsilon_0, \kappa_0, \mu_0).$$

Hence $U(x_0) \cap S_\zeta^>(\varepsilon_0, \kappa_0, \mu_0) \neq \varnothing$.

By Theorem 4.3, one has $S_\zeta^>(\cdot,\cdot,\cdot)$ is lsc at $(\varepsilon_0, \kappa_0, \mu_0)$. Thus there exists a neighborhood U' of $(\varepsilon_0, \kappa_0, \mu_0)$ such that

$$S_\zeta^>(\varepsilon, \kappa, \mu) \cap U(x_0) \neq \varnothing, \qquad \forall\, (\varepsilon, \kappa, \mu) \in U'.$$

Since $S_\zeta^>(\varepsilon, \kappa, \mu) \subset S(\varepsilon, \kappa, \mu)$ for every $(\varepsilon, \kappa, \mu)$, it follows that

$$S(\varepsilon, \kappa, \mu) \cap U(x_0) \neq \varnothing \qquad \forall\, (\varepsilon, \kappa, \mu) \in U',$$

which means $S(\cdot,\cdot,\cdot)$ is lsc at $(\varepsilon_0, \kappa_0, \mu_0)$. Therefore $S^w(\cdot,\cdot,\cdot)$ is also. □

Remark 4.6. *In this section, we have established continuity of approximate solution mappings with respect to the new metric. Our analysis shows that these results extend the continuity framework of Zhang and Xu [29], since they remain valid even when both the objective mapping and the feasible set are independently perturbed.*

5. Lipschitz properties of approximate solution maps

In this section we study the Lipschitz stability of approximate solution mappings. Using the metric G_e and scalarization techniques, we establish local Lipschitz continuity results that quantify the sensitivity of approximate solutions under perturbations of the objective and feasible set.

Theorem 5.6. *Let $\beta > 0$ and $(\kappa_0, \mu_0) \in \Lambda \times M$. Assume there are convex neighbourhoods $U(\kappa_0) \subset \Lambda$ and $U(\mu_0) \subset M$ and constants $q, L, M > 0$ such that:*

(i) For every $\kappa \in U(\kappa_0)$, $K(\kappa)$ has nonempty, convex and compact values.
(ii) For every $\mu \in U(\mu_0)$, the map $F(\cdot, \mu)$ has compact values and m_2 C–convex on $K(\kappa)$.
(iii) $F(y, \mu) \dot{-} F(x, \mu) \neq \varnothing$ for all $x, y \in K(\kappa_0)$ and all $\mu \in M$.
(iv) $K(\cdot)$ is q–Lipschitz on $U(\kappa_0)$: for all $\kappa_1, \kappa_2 \in U(\kappa_0)$,

$$K(\kappa_2) \;\subset\; K(\kappa_1) + q\,\|\kappa_1 - \kappa_2\|\,\mathbb{B}_X.$$

(v) For all $\mu_1, \mu_2 \in U(\mu_0)$, $g\,(F_{\mu_1}, F_{\mu_2}) \;\leq\; L\,\|\mu_1 - \mu_2\|$.
(vi) For all $\mu \in U(\mu_0)$ and $x_1, x_2 \in X$, $G_e\big(F(x_1, \mu), F(x_2, \mu)\big) \leq\; M\|x_1 - x_2\|$.

Then, for any $(\varepsilon_i, \kappa_i, \mu_i) \in (\beta, \infty) \times U(\kappa_0) \times U(\mu_0)$, $i = 1, 2$, the Hausdorff distance in X between the weak ε–minimal solution sets satisfies

$$\begin{aligned} &\mathcal{H}\big(S^w(\varepsilon_1, \kappa_1, \mu_1),\, S^w(\varepsilon_2, \kappa_2, \mu_2)\big) \\ &\qquad \leq\; \frac{\rho}{\beta}\,|\varepsilon_1 - \varepsilon_2| \;+\; \Big(q + \frac{2\rho q M}{\beta}\Big)\,\|\kappa_1 - \kappa_2\| \;+\; \frac{2\rho L}{\beta}\,\|\mu_1 - \mu_2\| \end{aligned} \tag{5.15}$$

where $\rho := \sup\{\|x - y\| : x, y \in K(\bar{\kappa})\} < \infty$. Moreover, the same bound holds for the ε–minimal solution sets,

$$\begin{aligned} &\mathcal{H}\big(S(\varepsilon_1, \kappa_1, \mu_1),\, S(\varepsilon_2, \kappa_2, \mu_2)\big) \\ &\qquad \leq\; \frac{\rho}{\beta}\,|\varepsilon_1 - \varepsilon_2| \;+\; \Big(q + \frac{2\rho q M}{\beta}\Big)\,\|\kappa_1 - \kappa_2\| \;+\; \frac{2\rho L}{\beta}\,\|\mu_1 - \mu_2\|. \end{aligned} \tag{5.16}$$

Proof. Since $S^w(\varepsilon, \kappa, \mu)$ coincides with $S_\zeta^w(\varepsilon, \kappa, \mu)$, the first conclusion follows directly from $S_\zeta^w(\varepsilon, \kappa, \mu)$. The proof of equation (5.15) is organized into three steps, as outlined below.

Step 1. Let $\varepsilon_1, \varepsilon_2 \in (\beta, \infty)$ and, without loss of generality, assume $\varepsilon_1 < \varepsilon_2$. Pick any $x_2 \in S_\zeta^w(\varepsilon_2, \bar{\kappa}, \bar{\mu})$ and choose $x_0 \in S_\zeta^w(0, \bar{\kappa}, \bar{\mu})$. Define

$$x_1 := \frac{\varepsilon_1}{\varepsilon_2} x_2 + \Big(1 - \frac{\varepsilon_1}{\varepsilon_2}\Big) x_0 \in K(\bar{\kappa}).$$

For any $y \in K(\bar{\kappa})$, concavity of $\zeta(y, \cdot, \bar{\mu})$ gives

$$\zeta(y, x_1, \bar{\mu}) \;\geq\; \frac{\varepsilon_1}{\varepsilon_2}\, \zeta(y, x_2, \bar{\mu}) + \Big(1 - \frac{\varepsilon_1}{\varepsilon_2}\Big)\, \zeta(y, x_0, \bar{\mu}) \;\geq\; \frac{\varepsilon_1}{\varepsilon_2}\, (-\varepsilon_2) + 0 \;=\; -\varepsilon_1.$$

Hence $x_1 \in S_\zeta^w(\varepsilon_1, \bar{\kappa}, \bar{\mu})$ and

$$\operatorname{dist}\big(x_2, S_\zeta^w(\varepsilon_1, \bar{\kappa}, \bar{\mu})\big) \;\leq\; \|x_2 - x_1\| \;=\; \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}\, \|x_2 - x_0\| \;\leq\; \frac{\rho}{\varepsilon_2}\, (\varepsilon_2 - \varepsilon_1) \;\leq\; \frac{\rho}{\beta}\, |\varepsilon_1 - \varepsilon_2|.$$

Taking the supremum over $x_2 \in S_\zeta^w(\varepsilon_2, \bar{\kappa}, \bar{\mu})$ yields

$$\sigma\big(S_\zeta^w(\varepsilon_2, \bar{\kappa}, \bar{\mu}), S_\zeta^w(\varepsilon_1, \bar{\kappa}, \bar{\mu})\big) \;\leq\; \frac{\rho}{\beta}\, |\varepsilon_1 - \varepsilon_2|.$$

Exchanging the roles of ε_1 and ε_2 gives the same bound for the reverse excess. Using $\mathcal{H}(A, B) = \max\{\sigma(A, B), \sigma(B, A)\}$ we obtain that

$$\mathcal{H}\left(S_\zeta^w(\varepsilon_1, \bar{\kappa}, \bar{\mu}), S_\zeta^w(\varepsilon_2, \bar{\kappa}, \bar{\mu})\right) \;\leq\; \frac{\rho}{\beta}\, |\varepsilon_1 - \varepsilon_2|. \tag{5.17}$$

Step 2. Fix $\bar{\kappa} \in U(\kappa_0)$ and choose $\varepsilon \in (\beta, \infty)$.

By Assumption (v) and Proposition 3.10, for all $x, y \in K(\bar{\kappa})$ we have

$$\begin{aligned}
\big|\zeta(y, x, \mu_1) - \zeta(y, x, \mu_2)\big| &= \big|I_e^{m_2}(F(y, \mu_1), F(x, \mu_1)) - I_e^{m_2}(F(y, \mu_2), F(x, \mu_2))\big| \\
&\leq \big|I_e^{m_2}(F(y, \mu_1), F(x, \mu_1)) - I_e^{m_2}(F(y, \mu_1), F(x, \mu_2))\big| \\
&\quad + \big|I_e^{m_2}(F(y, \mu_1), F(x, \mu_2)) - I_e^{m_2}(F(y, \mu_2), F(x, \mu_2))\big| \\
&\leq 2\, g(F_{\mu_1}, F_{\mu_2}) \;\leq\; 2L\, \|\mu_1 - \mu_2\|.
\end{aligned}$$

Hence, for any $v \in S_\zeta^w(\varepsilon, \bar{\kappa}, \mu_1)$,

$$\zeta(y, v, \mu_2) \;\geq\; \zeta(y, v, \mu_1) - 2L\|\mu_1 - \mu_2\| \;\geq\; -\varepsilon - 2L\|\mu_1 - \mu_2\| \qquad \forall\, y \in K(\bar{\kappa}).$$

Thus

$$S_\zeta^w(\varepsilon, \bar{\kappa}, \mu_1) \;\subset\; S_\zeta^w\big(\varepsilon + 2L\|\mu_1 - \mu_2\|, \bar{\kappa}, \mu_2\big). \tag{5.18}$$

Exchanging μ_1 and μ_2 yields the symmetric inclusion with the same tolerance.

Now fix $x_2 \in S_\zeta^w\big(\varepsilon + 2L\|\mu_1 - \mu_2\|, \bar{\kappa}, \mu_2\big)$. By definition,

$$\zeta(y, x_2, \mu_2) \;\geq\; -\big(\varepsilon + 2L\|\mu_1 - \mu_2\|\big) \qquad \forall\, y \in K(\bar{\kappa}).$$

Pick $x_0 \in S_\zeta^w(0, \bar{\kappa}, \mu_2)$ so that $\zeta(y, x_0, \mu_2) \geq 0$ for all $y \in K(\bar{\kappa})$. Let $\delta := 2L\|\mu_1 - \mu_2\|$ and define

$$\tau := \frac{\varepsilon}{\varepsilon + \delta} \in (0, 1], \qquad x_1 := \tau x_2 + (1 - \tau) x_0 \in K(\bar{\kappa}).$$

Using the concavity of $x \mapsto \zeta(y, x, \mu_2)$ on $K(\bar{\kappa})$,

$$\zeta(y, x_1, \mu_2) \;\geq\; \tau\, \zeta(y, x_2, \mu_2) + (1 - \tau)\, \zeta(y, x_0, \mu_2) \;\geq\; -\tau(\varepsilon + \delta) \;=\; -\varepsilon,$$

for every $y \in K(\bar{\kappa})$. Hence $x_1 \in S_\zeta^w(\varepsilon, \bar{\kappa}, \mu_2)$.

Therefore,

$$\operatorname{dist}\big(x_2, S_\zeta^w(\varepsilon, \bar{\kappa}, \mu_2)\big) \;\leq\; \|x_2 - x_1\| = (1 - \tau)\, \|x_2 - x_0\| = \frac{\delta}{\varepsilon + \delta}\, \|x_2 - x_0\| \leq \frac{\delta}{\varepsilon + \delta}\, \rho \leq \frac{\rho}{\beta}\, \delta,$$

where we used $\|x_2 - x_0\| \le \rho$ and $\varepsilon + \delta \ge \beta$. Taking the supremum over all such x_2 gives

$$\sigma\left(S_\zeta^w(\varepsilon + 2L\|\mu_1 - \mu_2\|, \bar\kappa, \mu_2), S_\zeta^w(\varepsilon, \bar\kappa, \mu_2)\right) \;\le\; \frac{2\rho L}{\beta}\,\|\mu_1 - \mu_2\|.$$

Finally, by the inclusion (5.18),

$$\sigma\left(S_\zeta^w(\varepsilon, \bar\kappa, \mu_1), S_\zeta^w(\varepsilon, \bar\kappa, \mu_2)\right) \;\le\; \frac{2\rho L}{\beta}\,\|\mu_1 - \mu_2\|.$$

The reverse excess bound follows by swapping μ_1 and μ_2, so the Hausdorff distance satisfies the same Lipschitz estimate. That is

$$\mathcal{H}\left(S_\zeta^w(\varepsilon, \bar\kappa, \mu_1), S_\zeta^w(\varepsilon, \bar\kappa, \mu_2)\right) \;\le\; \frac{2\rho L}{\beta}\,\|\mu_1 - \mu_2\|. \tag{5.19}$$

Step 3. Let $(\varepsilon, \bar\mu) \in (\beta, \infty) \times U(\mu_0)$ and $\kappa_1, \kappa_2 \in U(\kappa_0)$. We want to show that

$$\mathcal{H}\Big(S_\zeta^w(\varepsilon, \kappa_1, \bar\mu),\, S_\zeta^w(\varepsilon, \kappa_2, \bar\mu)\Big) \;\le\; \Big(q + \frac{2\rho M q}{\beta}\Big)\,\|\kappa_1 - \kappa_2\|. \tag{5.20}$$

Let $t := \varepsilon - \beta > 0$ and set

$$\upsilon := q\|\kappa_1 - \kappa_2\|, \qquad \delta_\kappa := 2M\upsilon = 2Mq\|\kappa_1 - \kappa_2\|.$$

By (iv), for each $x_1 \in K(\kappa_1)$ there exists $x_2 \in K(\kappa_2)$ with $\|x_2 - x_1\| \le \upsilon$, and for each $y \in K(\kappa_2)$ there exists $y_1 \in K(\kappa_1)$ with $\|y - y_1\| \le \upsilon$.

We distinguish two cases.

Case 1: $\upsilon \le \frac{t}{2M}$ (equivalently $\delta_\kappa \le t$). Pick any $x_1 \in S_\zeta^w(\varepsilon, \kappa_1, \bar\mu)$ and let $x_2 \in K(\kappa_2)$ be as above with $\|x_2 - x_1\| \le \upsilon$. For any $y \in K(\kappa_2)$ choose $y_1 \in K(\kappa_1)$ with $\|y - y_1\| \le \upsilon$. By definition of x_1, $\zeta(y_1, x_1, \bar\mu) \ge -\varepsilon$. Using (vi) together with Proposition 3.10 we obtain

$$\big|\zeta(y, x_2, \bar\mu) - \zeta(y_1, x_1, \bar\mu)\big| \;\le\; M\|y - y_1\| + M\|x_2 - x_1\| \;\le\; 2M\upsilon \;=\; \delta_\kappa.$$

Hence

$$\zeta(y, x_2, \bar\mu) \;\ge\; -\varepsilon - \delta_\kappa, \qquad \forall\, y \in K(\kappa_2),$$

which means

$$x_2 \in S_\zeta^w(\varepsilon + \delta_\kappa, \kappa_2, \bar\mu) \quad \text{and} \quad \operatorname{dist}\big(x_1, S_\zeta^w(\varepsilon + \delta_\kappa, \kappa_2, \bar\mu)\big) \le \|x_1 - x_2\| \le \upsilon.$$

Taking the supremum over $x_1 \in S_\zeta^w(\varepsilon, \kappa_1, \bar\mu)$ yields

$$\sigma\Big(S_\zeta^w(\varepsilon, \kappa_1, \bar\mu),\, S_\zeta^w(\varepsilon + \delta_\kappa, \kappa_2, \bar\mu)\Big) \;\le\; \upsilon \;=\; q\|\kappa_1 - \kappa_2\|. \tag{5.21}$$

Pick $u \in S_\zeta^w(\varepsilon + \delta_\kappa, \kappa_2, \bar\mu)$ and $x_0 \in S_\zeta^w(0, \kappa_2, \bar\mu)$. Set

$$\tau := \frac{\varepsilon}{\varepsilon + \delta_\kappa} \in (0, 1], \qquad x_1 := \tau u + (1 - \tau)x_0 \in K(\kappa_2).$$

By the concavity of $x \mapsto \zeta(y, x, \bar\mu)$ on $K(\kappa_2)$,

$$\zeta(y, x_1, \bar\mu) \;\ge\; \tau\,\zeta(y, u, \bar\mu) + (1 - \tau)\,\zeta(y, x_0, \bar\mu) \;\ge\; -\tau(\varepsilon + \delta_\kappa) + 0 \;=\; -\varepsilon,$$

hence $x_1 \in S_\zeta^w(\varepsilon, \kappa_2, \bar\mu)$, and

$$\operatorname{dist}\big(u, S_\zeta^w(\varepsilon, \kappa_2, \bar\mu)\big) \le \|u - x_1\| = (1 - \tau)\|u - x_0\| = \frac{\delta_\kappa}{\varepsilon + \delta_\kappa}\|u - x_0\| \le \frac{\delta_\kappa}{\beta}\,\rho.$$

Taking the supremum over u gives

$$\sigma\Big(S_\zeta^w(\varepsilon + \delta_\kappa, \kappa_2, \bar\mu),\, S_\zeta^w(\varepsilon, \kappa_2, \bar\mu)\Big) \;\le\; \frac{\rho}{\beta}\,\delta_\kappa \;=\; \frac{2\rho M q}{\beta}\,\|\kappa_1 - \kappa_2\|. \tag{5.22}$$

Combining (5.21) and (5.22) by the triangle inequality,

$$\sigma\Big(S_\zeta^w(\varepsilon,\kappa_1,\bar{\mu}),\, S_\zeta^w(\varepsilon,\kappa_2,\bar{\mu})\Big) \;\le\; \Big(q+\frac{2\rho Mq}{\beta}\Big)\,\|\kappa_1-\kappa_2\|.$$

Interchanging κ_1 and κ_2 yields the same bound for the reverse excess, and hence the Hausdorff distance satisfies (5.20).

Case 2: $\eta > \frac{t}{2M}$ (equivalently $\delta_\kappa > t$). Choose $n \in \mathbb{N}$ large enough so that

$$\eta_i := q\Big\|\kappa^{(i)}-\kappa^{(i-1)}\Big\| \le \frac{t}{2M}, \qquad \kappa^{(i)} := \kappa_1 + \frac{i}{n}(\kappa_2-\kappa_1), \quad i=0,1,\dots,n.$$

Applying Case 1 to each consecutive pair $(\kappa^{(i-1)},\kappa^{(i)})$ yields

$$\mathcal{H}\Big(S_\zeta^w(\varepsilon,\kappa^{(i-1)},\bar{\mu}),\, S_\zeta^w(\varepsilon,\kappa^{(i)},\bar{\mu})\Big) \;\le\; \Big(q+\frac{2\rho Mq}{\beta}\Big)\,\Big\|\kappa^{(i)}-\kappa^{(i-1)}\Big\|.$$

Summing and using the triangle inequality for $\mathcal{H}$ gives

$$\mathcal{H}\Big(S_\zeta^w(\varepsilon,\kappa_1,\bar{\mu}),\, S_\zeta^w(\varepsilon,\kappa_2,\bar{\mu})\Big) \;\le\; \Big(q+\frac{2\rho Mq}{\beta}\Big)\,\|\kappa_1-\kappa_2\|.$$

Combining Steps (5.17), (5.19), and (5.20), we obtain equation (5.15).

It follows from Lemma 4.4) that

$$\mathrm{cl}\,(S(\varepsilon,\kappa,\mu)) \;=\; S_\zeta^w(\varepsilon,\kappa,\mu).$$

We obtain, for any $(\varepsilon_i,\kappa_i,\mu_i)$ $(i=1,2)$,

$$\mathcal{H}\big(S(\varepsilon_1,\kappa_1,\mu_1),\, S(\varepsilon_2,\kappa_2,\mu_2)\big) = \mathcal{H}\big(S_\zeta^w(\varepsilon_1,\kappa_1,\mu_1),\, S_\zeta^w(\varepsilon_2,\kappa_2,\mu_2)\big).$$

Therefore, equation (5.16) follows, completing the proof.

□

Remark 5.7. *We emphasize that the contributions of this section are distinct from existing results in the literature. Our analysis focuses on the Lipschitz behavior of the approximate minimal and approximate weak minimal solution mappings for parametric set optimization problems under the order relation induced by the Minkowski difference. In contrast, previous studies have mainly investigated qualitative stability properties such as semicontinuity, well-posedness, and convergence of minimal and weak minimal solution mappings under the same order relation.*

We illustrate Theorem 5.6 with the following example, which verifies its assumptions step by step and demonstrates how the general Lipschitz stability result applies in a concrete setting.

Example 5.4. *Let $X = Y = \mathbb{R}$, let $C = [0,\infty)$ and fix $e = 1 \in \mathrm{int}(C)$. For $(\kappa,\mu) \in [0,1]\times[0,1]$, set*

$$K(\kappa) := [\kappa-1, \kappa+1].$$

Fix $r > 0$ and define the set–valued map

$$F(x,\mu) := [\, f(x,\mu)-r,\; f(x,\mu)+r\,], \qquad f(x,\mu) := (x-\mu)^2.$$

It is clear that $K(\cdot)$ has convex and compact values and is 1–Lipschitz on $[0,1]$, while $F(x,\mu)$ has compact values on $K(\kappa)$. We now verify assumptions (iii), (v)–(vi) in Theorem 5.6.

(iii). *For $\alpha \in [0,1]$ and $x_1,x_2 \in K(\kappa)$,*

$$F(\alpha x_1+(1-\alpha)x_2,\mu) = [\, f(\alpha x_1+(1-\alpha)x_2,\mu)-r,\; f(\alpha x_1+(1-\alpha)x_2,\mu)+r\,],$$

and

$$\alpha F(x_1,\mu)+(1-\alpha)F(x_2,\mu) = [\,\alpha f(x_1,\mu)+(1-\alpha)f(x_2,\mu)-r,\; \alpha f(x_1,\mu)+(1-\alpha)f(x_2,\mu)+r\,].$$

Since $x \mapsto f(x,\mu)$ is convex, we have

$$f(\alpha x_1 + (1-\alpha)x_2, \mu) \le \alpha f(x_1,\mu) + (1-\alpha) f(x_2,\mu),$$

which implies

$$F(\alpha x_1 + (1-\alpha)x_2, \mu) \preceq_C^{m_2} \alpha F(x_1,\mu) + (1-\alpha)F(x_2,\mu).$$

Thus $F(\cdot,\mu)$ is m_2 C–convex on $K(\kappa)$.

(v). *We compute*

$$G_e\big(F(x,\mu_1), F(x,\mu_2)\big) = \big|f(x,\mu_1) - f(x,\mu_2)\big| = \big|(\mu_1-\mu_2)(\mu_1+\mu_2-2x)\big|.$$

Let

$$R := \sup\{|x| : x \in K(\kappa),\ \kappa \in [0,1]\} = 2, \qquad M_\mu := \sup\{|\mu| : \mu \in [0,1]\} = 1.$$

Hence

$$G_e\big(F(x,\mu_1), F(x,\mu_2)\big) \le 2(R+M_\mu)\,|\mu_1-\mu_2| = 6\,|\mu_1-\mu_2|,$$

so that

$$g(F_{\mu_1}, F_{\mu_2}) \le 6\,|\mu_1-\mu_2|.$$

(vi). *Similarly,*

$$G_e\big(F(x_1,\mu), F(x_2,\mu)\big) = \big|f(x_1,\mu) - f(x_2,\mu)\big| = \big|(x_1-x_2)(x_1+x_2-2\mu)\big| \le 6\,|x_1-x_2|.$$

Therefore, all assumptions of Theorem 5.6 are satisfied with $q=1$, $\rho=2$, $L=6$, and $M=6$. In particular, for $(\varepsilon_0,\kappa_0,\mu_0) \in [1,\infty)\times[0,1]\times[0,1]$, Theorem 5.6 yields the estimates

$$\mathcal{H}(S^w(\varepsilon_1,\kappa_1,\mu_1),\ S^w(\varepsilon_2,\kappa_2,\mu_2)) \le 2\,|\varepsilon_1-\varepsilon_2| + 24\,|\mu_1-\mu_2| + 25\,|\kappa_1-\kappa_2|,$$

and

$$\mathcal{H}(S(\varepsilon_1,\kappa_1,\mu_1),\ S(\varepsilon_2,\kappa_2,\mu_2)) \le 2\,|\varepsilon_1-\varepsilon_2| + 24\,|\mu_1-\mu_2| + 25\,|\kappa_1-\kappa_2|.$$

6. Conclusions

The central contribution of this paper is to advance the stability analysis of approximate weak solutions in set optimization under the m_2-order. Building on earlier continuity and convexity results for nonlinear scalarization functions, we extend the framework to settings in which both the feasible set and the objective mapping are subject to perturbations. Through the proposed scalarizing function, we establish continuity of the approximate minimal and approximate weak minimal solution mappings. In addition, we derive quantitative stability results by proving Lipschitz properties of these mappings. Collectively, our findings yield the first explicit stability bounds for approximate weak solutions with respect to the m_2-order.

Acknowledgments

The first author gratefully acknowledges the financial support of Pibulsongkram Rajabhat University (PSRU), Thailand Science Research and Innovation (TSRI), and the National Science, Research and Innovation Fund (NSRF). The second author was partially supported by the Frontier Research and Innovation Cluster Fund, Naresuan University (Grant No. R2569C008).

REFERENCES

[1] Anh, L. Q.; Danh, N. H.; Tam, T. N. Continuity of solution maps to parametric set optimization problems via parametric equilibrium problems. *Acta Math. Vietnam.* **45** (2020), no. 2, 383–395.

[2] Anh, L. Q.; Duy, T. Q.; Hien, D. V. Stability of efficient solutions to set optimization problems. *J. Global Optim.* **78** (2020), no. 3, 563–580.

[3] Anh, L. Q.; Duy, T. Q.; Hien, D. V.; Kuroiwa, D.; Petrot, N. Convergence of solutions to set optimization problems with the set less order relation. *J. Optim. Theory Appl.* **185** (2020), no. 2, 416–432.

[4] Anh, L. Q.; Danh, N. H.; Duoc, P. T.; Tam, T. N. Qualitative properties of solutions to set optimization problems. *Comput. Appl. Math.* **40** (2021), no. 66, 20 pp.

[5] Ansari, Q. H.; Hussain, N.; Sharma, P. K. Convergence of the solution sets for set optimization problems. *J. Nonlinear Var. Anal.* **6** (2022), no. 3, 165–183.

[6] Bao, T. Q.; Mordukhovich, B. S. Set-valued optimization in welfare economics. *Adv. Math. Econ.* **13** (2010), 113–153.

[7] Chen, G. Y.; Huang, X. X.; Yang, X. Q. *Vector optimization, set-valued and variational analysis*. Springer, Heidelberg/Berlin, 2005.

[8] Göpfert, A.; Riahi, H.; Tammer, C.; Zălinescu, C. *Variational methods in partially ordered spaces*. Springer, Berlin, 2003.

[9] Gupta, M.; Srivastava, M. Approximate solutions and Levitin–Polyak well-posedness for set optimization using weak efficiency. *J. Optim. Theory Appl.* **186** (2020), 191–208.

[10] Gupta, S.; Gupta, R.; Srivastava, M. On scalarization and well-posedness in set optimization with a partial set order relation. *Positivity* **28** (2024), no. 1, 15 pp.

[11] Hamel, A. H.; Heyde, F.; Löhne, A.; Rudloff, B.; Schrage, C., eds. *Set optimization and applications—The state of the art*. Springer, New York, NY, 2015.

[12] Han, Y.; Huang, N. J. Well-posedness and stability of solutions for set optimization problems. *Optimization* **66** (2016), no. 1, 17–33.

[13] Han, Y. A Hausdorff-type distance, the Clarke generalized directional derivative and applications in set optimization problems. *Appl. Anal.* **99** (2020), no. 11, 1860–1880.

[14] Han, Y.; Li, S.-J. Stability of the approximate solution sets for set optimization problems with the perturbations of feasible set and objective mapping. *Optimization* **73** (2023), no. 13, 3755–3785.

[15] Jahn, J. *Vector optimization: Theory, applications and extensions*. 2nd ed. Springer, Heidelberg/Berlin, 2011.

[16] Jahn, J.; Ha, T. X. D. New order relations in set optimization. *J. Optim. Theory Appl.* **148** (2001), 209–236.

[17] Kapoor, M.; Ahmad, W.; Sharma, P. K. Path connectedness of solution sets for partially ordered set optimization problems. *J. Nonlinear Var. Anal.* **8** (2024), no. 6, 973–986.

[18] Karaman, E.; Soyertem, M.; Atasever Güvenç, I.; Tozkan, D.; Küçük, M.; Küçük, Y. Partial order relations on family of sets and scalarizations for set optimization. *Positivity* **22** (2018), no. 3, 783–802.

[19] Karuna; Lalitha, C. S. Convergence, scalarization and continuity of minimal solutions in set optimization. *J. Oper. Res. Soc. China* **12** (2024), 773–793.

[20] Khan, A. A.; Tammer, C.; Zălinescu, C. *Set-valued optimization: An introduction with applications*. Springer, Heidelberg/Berlin, 2015.

[21] Khoshkhabar-Amiranloo, S. Stability of minimal solutions to parametric set optimization problems. *Appl. Anal.* **97** (2018), no. 14, 2510–2522.

[22] Kuroiwa, D. The natural criteria in set-valued optimization. *RIMS Kokyuroku Kyoto Univ.* **1031** (1998), 85–90.

[23] Kuroiwa, D. On set-valued optimization. *Nonlinear Anal.* **47** (2001), 1395–1400.

[24] Löhne, A. *Vector optimization with infimum and supremum*. Springer, Berlin, 2011.

[25] Preechasilp, P.; Wangkeeree, R. A note on semicontinuity of the solution mapping for parametric set optimization problems. *Optim. Lett.* **13** (2019), no. 5, 1085–1094.

[26] Preechasilp, P.; Wangkeeree, R. Pointwise well-posedness and scalarization for set optimization problems with a partial order relation. *Thai J. Math.* **18** (2020), no. 3, 1623–1638.

[27] Sion, M. On general minimax theorems. *Pacific J. Math.* **8** (1958), no. 1, 171–176.

[28] Zălinescu, C. *Convex analysis in general vector spaces*. World Scientific Publishing, Singapore, 2002.

[29] Zhang, Z. R.; Xu, Y. D. The continuity and convexity of a nonlinear scalarization function with applications in set optimization problems involving a partial order relation. *Mathematics* **12** (2024), no. 23, 22 pp.

[30] Zhao, P.; Xu, Y.; Huang, B. Approximate solutions for set optimization with an order cone that has nonempty quasirelative interiors. *J. Inequal. Appl.* **2022** (2022), no. 101, 15 pp.

[31] Zhou, Z.; Feng, K.; Ansari, Q. H. Well-posedness of set optimization problems with set order defined by Minkowski difference. *J. Optim. Theory Appl.* **204** (2025), no. 2, 31–52.

[1] PROGRAM IN MATHEMATICS, FACULTY OF EDUCATION, PIBULSONGKRAM RAJABHAT UNIVERSITY, PHITSANULOK 65000, THAILAND
Email address: E-mail pakkapon.p@psru.ac.th

[2] DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PHITSANULOK 65000, THAILAND
Email address: E-mail rabianw@nu.ac.th